Lesson 4–7

Example 1 Verify Inverse Matrices

Determine whether each pair of matrices are inverses.

a. $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Find $A \cdot B$.

$$A \cdot B = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & -\frac{1}{2}+\frac{1}{2} \\ -\frac{1}{2}+\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Matrix multiplication

Now find $B \cdot A$.

$$B \cdot A = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & -\frac{1}{2}+\frac{1}{2} \\ -\frac{1}{2}+\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Matrix multiplication

Since $A \cdot B = B \cdot A = I$, A and B are inverses.

b. $C = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$ **and** $D = \begin{bmatrix} 2 & -1 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$

Find $C \cdot D$.

$$C \cdot D = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} -2+3 & 1+1 \\ -6+6 & 3+2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$$ Matrix multiplication

Since $C \cdot D \neq I$, they are *not* inverses.

Example 2 Find the Inverse of a Matrix

Find the inverse of each matrix, if it exists.

a. $P = \begin{bmatrix} 7 & -1 \\ 5 & 4 \end{bmatrix}$

Find the value of the determinant.

$\begin{vmatrix} 7 & -1 \\ 5 & 4 \end{vmatrix} = 28 - (-5) = 33$

Since the determinant does not equal 0, P^{-1} exists.

$$P^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ Definition of inverse

$$= \frac{1}{7(4) - (-1)(5)}\begin{bmatrix} 4 & 1 \\ -5 & 7 \end{bmatrix}$$ $a = 7, b = -1, c = 5, d = 4$

$$= \frac{1}{33}\begin{bmatrix} 4 & 1 \\ -5 & 7 \end{bmatrix} \text{ or } \begin{bmatrix} \frac{4}{33} & \frac{1}{33} \\ -\frac{5}{33} & \frac{7}{33} \end{bmatrix}$$ Simplify.

Check $\begin{bmatrix} 7 & -1 \\ 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} \frac{4}{33} & \frac{1}{33} \\ -\frac{5}{33} & \frac{7}{33} \end{bmatrix} = \begin{bmatrix} \frac{28}{33} + \frac{5}{33} & \frac{7}{33} + \left(-\frac{7}{33}\right) \\ \frac{20}{33} + \left(-\frac{20}{33}\right) & \frac{5}{33} + \frac{28}{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

b. $Z = \begin{bmatrix} 10 & 4 \\ -5 & -2 \end{bmatrix}$

Find the value of the determinant.

$\begin{vmatrix} 10 & 4 \\ -5 & -2 \end{vmatrix} = -20 - (-20) = 0$

Since the determinant equals 0, Z^{-1} does not exist.

Example 3 Use Inverses to Solve a Problem

a. CRYPTOGRAPHY Use the table below to assign a number to each letter in the message RING_FOUR_TIMES. Then code the message with the matrix $B = \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}$.

CODE								
__ 0	**C** 5	**F** 11	**I** 17	**L** 23	**O** 4	**R** 10	**U** 16	**X** 22
A 1	**D** 7	**G** 13	**J** 19	**M** 25	**P** 6	**S** 12	**V** 18	**Y** 24
B 3	**E** 9	**H** 15	**K** 21	**N** 2	**Q** 8	**T** 14	**W** 20	**Z** 26

Convert the message to numbers using the table. Since there are an odd number of letters to be coded, add a 0 at the end of the message.

R	I	N	G	_	F	O	U	R	_	T	I	M	E	S	
10	17	2	13	0	11	4	16	10	0	14	17	25	9	12	0

Write the message in matrix form. Then multiply the message matrix D by the coding matrix B.

$$DB = \begin{bmatrix} 10 & 17 \\ 2 & 13 \\ 0 & 11 \\ 4 & 16 \\ 10 & 0 \\ 14 & 17 \\ 25 & 9 \\ 12 & 0 \end{bmatrix} \cdot \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}$$

Write an equation. Be sure to include the 0 at the end of the message.

$$= \begin{bmatrix} -20+17 & -10+34 \\ -4+13 & -2+26 \\ 0+11 & 0+22 \\ -8+16 & -4+32 \\ -20+0 & -10+0 \\ -28+17 & -14+34 \\ -50+9 & -25+18 \\ -24+0 & -12+0 \end{bmatrix}$$

Matrix multiplication.

$$= \begin{bmatrix} -3 & 24 \\ 9 & 24 \\ 11 & 22 \\ 8 & 28 \\ -20 & -10 \\ -11 & 20 \\ -41 & -7 \\ -24 & -12 \end{bmatrix}$$

Simplify.

The coded message is –3 | 24 | 9 | 24 | 11 | 22| 8 | 28 | –20 | –10 | –11 | 20 | –41 | –7 | –24 | –12 .

b. Use the inverse matrix B^{-1} to decode the message in Example 3a.

First find the inverse matrix of $B = \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}$.

$$B^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ Definition of inverse

$$= \frac{1}{-2(2) - (-1)(1)}\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$$ $a = -2$, $b = -1$, $c = 1$, $d = 2$

$$= -\frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \text{ or } \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$ Simplify.

Next, decode the message by multiplying the coded matrix C by B^{-1}.

$$CB^{-1} = \begin{bmatrix} -3 & 24 \\ 9 & 24 \\ 11 & 22 \\ 8 & 28 \\ -20 & -10 \\ -11 & 20 \\ -41 & -7 \\ -24 & -12 \end{bmatrix} \cdot \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2+8 & 1+16 \\ -6+8 & -3+16 \\ -\frac{22}{3}+\frac{22}{3} & -\frac{11}{3}+\frac{44}{3} \\ -\frac{16}{3}+\frac{28}{3} & -\frac{8}{3}+\frac{56}{3} \\ \frac{40}{3}+\left(-\frac{10}{3}\right) & \frac{20}{3}+\left(-\frac{20}{3}\right) \\ \frac{22}{3}+\frac{20}{3} & \frac{11}{3}+\frac{40}{3} \\ \frac{82}{3}+\left(-\frac{7}{3}\right) & \frac{41}{3}+\left(-\frac{14}{3}\right) \\ 16+(-4) & \frac{24}{3}+\left(-\frac{24}{3}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 17 \\ 2 & 13 \\ 0 & 11 \\ 4 & 16 \\ 10 & 0 \\ 14 & 17 \\ 25 & 9 \\ 12 & 0 \end{bmatrix}$$

Use the table again to convert the numbers to letters. You can now read the message.

10	17	2	13	0	11	4	16	10	0	14	17	25	9	12	0
R	I	N	G	_	F	O	U	R	_	T	I	M	E	S	