Lesson 10–6

Example 1 Exponential Decay of the Form $y = a(1 - r)^t$.

RECREATION A particular chemical must be added to a swimming pool at regular intervals because it is released from the water into the air at the rate of 5% per hour. Sixteen ounces of the chemical is added at 8 am. At what time will three–fourths of this chemical be gone from the pool?

Explore Read the problem. The problem gives the amount of chemical added to the pool and the rate at which the chemical is eliminated. It asks you to find the time it will take for three–fourths of the original amount of chemical to be eliminated from the pool. You then must add that time to 8 am to find the time of day.

Plan Use the formula $y = a(1 - r)^t$. Let t be the number of hours since adding the chemical. The initial amount a is 16 ounces, and the percent of decrease r is 0.05. The amount remaining y is one–fourth of 16 or 4.

Solve

$y = a(1 - r)^t$	Exponential decay formula
$4 = 16(1 - 0.05)^t$	Replace y with 4, a with 16, and r with 0.05.
$0.25 = (0.95)^t$	Divide each side by 16.
$\log 0.25 = \log (0.95)^t$	Property of Equality for Logarithms
$\log 0.25 = t \log (0.95)$	Product Property for Logarithms
$\frac{\log 0.25}{\log 0.95} = t$	Divide each side by log 0.95.
$27.0268 \approx t$	Use a calculator.

Since $27.0268 \approx 27$, it will take approximately 27 hours for three–fourths of the chemical to be eliminated. The time will be 11 am of the next day.

Examine Use the formula to find how much of the original 16 ounces of chemical will remain after 27 hours.

$y = a(1 - r)^t$	Exponential decay formula
$y = 16(1 - 0.05)^{27}$	Replace a with 16, r with 0.05, and t with 27.
$y \approx 4.0055$	Use a calculator.

One–fourth of 16 is 4, so the answer seems reasonable.

Example 2 Exponential Decay of the Form $y = ae^{-kt}$.

CHEMISTRY The *half–life* of a radioactive substance is the time it takes for half of the atoms of the substance to decay. Each element has a unique half–life. Radon–222 has a half–life of about 3.8 days, while thorium–234 has a half–life of about 24 days. Find the value of *k* for each element and compare their equations for decay.

The equations will be of the form $y = ae^{-kt}$, where t is in days. To determine the constant k for each element, let a be the initial amount of the substance. The amount y that remains after t days of the half–life is then represented by $0.5a$. Use this idea to find the value of k for each element and then to write their equations.

Radon–222

$y = ae^{-kt}$	Exponential decay formula
$0.5a = ae^{-k(3.8)}$	Replace y with $0.5a$ and t with 3.8.
$0.5 = e^{-3.8k}$	Divide each side by a.
$\ln 0.5 = \ln e^{-3.8k}$	Property of Equality for Logarithmic Functions
$\ln 0.5 = -3.8k$	Inverse Property of Exponents and Logarithms
$\frac{\ln 0.5}{-3.8} = k$	Divide each side by –3.8.
$0.1824 \approx k$	Use a calculator.

Thorium–234

$y = ae^{-kt}$	Exponential decay formula
$0.5a = ae^{-k(24)}$	Replace y with $0.5a$ and t with 24.
$0.5 = e^{-24k}$	Divide each side by a.
$\ln 0.5 = \ln e^{-24k}$	Property of Equality for Logarithmic Functions
$\ln 0.5 = -24k$	Inverse Property of Exponents and Logarithms
$\frac{\ln 0.5}{-24} = k$	Divide each side by –24.
$0.0289 \approx k$	Use a calculator.

The equations for radon–222 and thorium–234 are $y = ae^{-0.1824t}$ and $y = ae^{-0.0289t}$, respectively. For both equations, t represents time in days. In comparing the equations, it appears that the longer the half–life, the smaller the value of k.

Example 3 Exponential Growth of the Form $y = a(1 + r)^t$.

Multiple–Choice Test Item

In 1980, the value of farming land in a region of Wyoming was \$150 per acre. Since then, the value has increased by exactly 0.75% per year. If the land continues to increase in value at this rate, what will the approximate value of the land per acre be in 2005?

A. \$610 **B.** \$330 **C.** \$181
D. \$175 **E.** \$1098

Read the Test Item

You need to find the value of land 2005 – 1980 or 25 years later. Since the land is increasing in value at a fixed percent each year, use the formula $y = a(1 + r)^t$.

Solve the Test Item

The initial value a is 150, the percent of increase is 0.75% or 0.0075, and the time t is 25 years.

$y = a(1 + r)^t$	Exponential growth formula
$y = 150(1 + 0.0075)^{25}$	Replace a with 150, r with 0.0075, and t with 25.
$y = 150(1.0075)^{25}$	Simplify.
$y \approx 180.81$	Use a calculator.

The answer is C.

Example 4 **Exponential Growth of the Form $y = ae^{kt}$.**

SAVINGS Sue invests $1000 at 5% interest compounded continuously and Norma invests $1250 at 3.5% interest compounded continuously. When interest is compounded continuously, the amount *A* in an account after *t* years is found using the formula $A = Pe^{rt}$, where *P* is the amount of principal and *r* is the annual interest rate. In how many years will Sue's account be greater than Norma's account?

Since Sue's interest rate is greater than Norma's, it seem likely that Sue's account will eventually be greater than Norma's. You need to write a function for Sue's account and for Norma's account using the formula and then write an inequality. Let *S* be the amount in Sue's account and *N* be the amount in Norma's account.

$A = Pe^{rt}$		$A = Pe^{rt}$	
$S = 1000e^{0.05t}$	Sue's principal is 1000 and rate is 5% or 0.05.	$N = 1250e^{0.035t}$	Norma's principal is 1250 and rate is 3.5% or 0.035.

You want to find *t* such that $S > N$.

$1000e^{0.05t} > 1250e^{0.035t}$	$S > N$
$\ln 1000e^{0.05t} > \ln 1250e^{0.035t}$	Property of Inequality for Logarithms
$\ln 1000 + \ln e^{0.05t} > \ln 1250 + \ln e^{0.035t}$	Product Property of Logarithms
$\ln 1000 + 0.05t > \ln 1250 + 0.035t$	Inverse Property of Exponents and Logarithms
$\ln 1000 + 0.015t > \ln 1250$	Subtract $0.035t$ from each side.
$0.015t > \ln 1250 - \ln 1000$	Subtract ln 1000 from each side.
$t > \dfrac{\ln 1250 - \ln 1000}{0.015}$	Divide each side by 0.015.
$t \geq 14.88$	Use a calculator.

After about 15 years, Sue's account will be greater than Norma's account.