Lesson 11–8

Example 1 **Summation Formula**

Prove that the sum of the cubes of the first n positive integers is $\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$. That is, prove that $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$.

Step 1 When $n = 1$, the left side of the given equation is 1^3 or 1. The right side is $\frac{1^4}{4}+\frac{1^3}{2}+\frac{1^2}{4}$ or 1. Thus, the equation is true for $n = 1$.

Step 2 Assume that $1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^4}{4} + \frac{k^3}{2} + \frac{k^2}{4}$ for some positive integer k.

Step 3 Show that the given equation is true for $n = k + 1$.

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3$

$= \frac{k^4}{4} + \frac{k^3}{2} + \frac{k^2}{4} + (k+1)^3$ Add $(k + 1)^3$ to each side.

$= \frac{k^4 + 2k^3 + k^2 + 4(k+1)^3}{4}$ Add.

$= \frac{k^4 + 2k^3 + k^2 + 4k^3 + 12k^2 + 12k + 4}{4}$ Multiply.

$= \frac{(k^4 + 4k^3 + 6k^2 + 4k + 1) + 2(k^3 + 3k^2 + 3k + 1) + (k^2 + 2k + 1)}{4}$ Group terms.

$= \frac{(k+1)^4 + 2(k+1)^3 + (k+1)^2}{4}$ Factor.

$= \frac{(k+1)^4}{4} + \frac{(k+1)^3}{2} + \frac{(k+1)^2}{4}$ Rewrite.

The last expression above is the right side of the equation to be proved, where n has been replaced by $k + 1$. Thus, the equation is true for $n = k + 1$.

This proves that $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$ for all positive integers n.

Example 2 **Divisibility**

Prove that $7^n + 5$ is divisible by 6 for all positive integers n.

Step 1 When $n = 1$, $7^n + 5 = 7^1 + 5$ or 12. Since 12 is divisible by 6, the statement is true for $n = 1$.

Step 2 Assume that $7^k + 5$ is divisible by 6 for some positive integer k. This means that there is a whole number r such that $7^k + 5 = 6r$.

Step 3 Show that the statement is true for $n = k + 1$.

$7^k + 5 = 6r$	Inductive hypothesis
$7^k = 6r - 5$	Subtract 5 from each side.
$7(7^k) = 7(6r - 5)$	Multiply each side by 7.
$7^{k+1} = 42r - 35$	Simplify.
$7^{k+1} + 5 = 42r - 30$	Add 5 to each side.
$7^{k+1} + 5 = 6(7r - 5)$	Factor.

Since r is a whole number, $7r - 5$ is a whole number. Therefore, $7^{k+1} + 5$ is divisible by 6. Thus, the statement is true for $n = k + 1$.

This proves that $7^n + 5$ is divisible by 6 for all positive integers n.

Example 3 **Counterexample**

Find a counterexample for the formula $1^3 + 2^3 + 3^3 + \ldots + n^3 = 1 + (8n - 8)$.

Check the first few positive integers.

n	**Left Side of Formula**	**Right Side of Formula**	
1	1^3 or 1	$1 + [8(1) - 8] = 1 + 0 = 1$	true
2	$1^3 + 2^3 = 1 + 8 = 9$	$1 + [8(2) - 8] = 1 + 8 = 9$	true
3	$1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$	$1 + [8(3) - 8] = 1 + 16 = 17$	false

The value of $n = 3$ is a counterexample for the formula.