Lesson 4–3

Example 1 Dimensions of Matrix Products

Determine whether each matrix product is defined. If so, state the dimensions of the product.

a. $A_{3\times 2}$ **and** $B_{3\times 2}$

$$\begin{matrix} A & \cdot & B \\ 3\times 2 & & 3\times 2 \end{matrix}$$

The inner dimensions are not equal, so the matrix product is not defined.

b. $A_{1\times 5}$ **and** $B_{5\times 1}$

$$\begin{matrix} A & \cdot & B \\ 1\times 5 & & 5\times 1 \end{matrix}$$

The inner dimensions are equal so the matrix product is defined. The dimensions of the product are 1×1.

Example 2 Multiply Square Matrices

Find EF **if** $E = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & -2 \\ -1 & -3 & 2 \end{bmatrix}$ **and** $F = \begin{bmatrix} -2 & 3 & 1 \\ 0 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix}$.

$$EF = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & -2 \\ -1 & -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & 1 \\ 0 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix}$$

Step 1 Multiply the first row in *E* and the first column in *F*.

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & -2 \\ -1 & -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & 1 \\ 0 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix} = \begin{bmatrix} -1(-2)+2(0)+0(-5) & & \\ & & \\ & & \end{bmatrix}$$

Step 2 Simplify the entry from Step 1. Then, multiply the first row in *E* and the second column in *F*.

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & -2 \\ -1 & -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & 1 \\ 0 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -1(3)+2(-1)+0(0) & \\ & & \\ & & \end{bmatrix}$$

Step 3 Simplify the entry from Step 2. Then, multiply the first row in *E* and the third column in *F*.

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & -2 \\ -1 & -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & 1 \\ 0 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -5 & -1(1)+2(1)+0(-3) \\ & & \\ & & \end{bmatrix}$$

Step 4 Simplify the entry from Step 3. Then, multiply the second row in E and the first column in F.

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & -2 \\ -1 & -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & 1 \\ 0 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 1 \\ 1(-2)+3(0)+(-2)(-5) & & \\ & & \end{bmatrix}$$

Step 5 Simplify the entry from Step 4. Then, multiply the second row in E and the second column in F.

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & -2 \\ -1 & -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & 1 \\ 0 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 1 \\ 8 & 1(3)+3(-1)+(-2)(0) & \\ & & \end{bmatrix}$$

Step 6 Simplify the entry from Step 5. Then, multiply the second row in E and the third column in F.

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & -2 \\ -1 & -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & 1 \\ 0 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 1 \\ 8 & 0 & 1(1)+3(1)+(-2)(-3) \\ & & \end{bmatrix}$$

Step 7 Simplify the entry from Step 6. Then, multiply the third row in E and the first column in F.

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & -2 \\ -1 & -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & 1 \\ 0 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 1 \\ 8 & 0 & 10 \\ (-1)(-2)+(-3)(0)+2(-5) & & \end{bmatrix}$$

Step 8 Simplify the entry from Step 7. Then, multiply the third row in E and the second column in F.

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & -2 \\ -1 & -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & 1 \\ 0 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 1 \\ 8 & 0 & 10 \\ -8 & -1(3)+(-3)(-1)+2(0) & \end{bmatrix}$$

Step 9 Simplify the entry from Step 8. Then, multiply the third row in E and the third column in F.

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & -2 \\ -1 & -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & 1 \\ 0 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 1 \\ 8 & 0 & 10 \\ -8 & 0 & -1(1)+(-3)(1)+2(-3) \end{bmatrix}$$

Step 10 Simplify the entry from Step 9.

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & -2 \\ -1 & -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 & 1 \\ 0 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 1 \\ 8 & 0 & 10 \\ -8 & 0 & -10 \end{bmatrix}$$

So, $EF = \begin{bmatrix} 2 & -5 & 1 \\ 8 & 0 & 10 \\ -8 & 0 & -10 \end{bmatrix}$.

Example 3 Multiply Matrices with Different Dimensions

OLYMPICS The table shows the number of each type of medal won by the top five countries in the 2000 Summer Olympics. A gold medal is worth 3 points, a silver is worth 2 points, and a bronze is worth 1 point. Find the total number of points scored by each country.

Country	Gold	Silver	Bronze
United States	40	24	33
Russia	32	28	28
China	28	16	15
Australia	16	25	17
Germany	13	17	26

Source: 2001 ESPN Sports Almanac

Explore The total number of points scored by each country can be found by multiplying the number of medals of each type by the points awarded for each type of medal.

Plan Write the number of medals and the points for each type in matrix form. Set up the matrices so that the number of rows in the Points matrix equals the number of columns in the Medals matrix.

Medals **Points**

$$M = \begin{bmatrix} 40 & 24 & 33 \\ 32 & 28 & 28 \\ 28 & 16 & 15 \\ 16 & 25 & 17 \\ 13 & 17 & 26 \end{bmatrix} \qquad P = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Solve Multiply the matrices.

$$MP = \begin{bmatrix} 40 & 24 & 33 \\ 32 & 28 & 28 \\ 28 & 16 & 15 \\ 16 & 25 & 17 \\ 13 & 17 & 26 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 40(3)+24(2)+33(1) \\ 32(3)+28(2)+28(1) \\ 28(3)+16(2)+15(1) \\ 16(3)+25(2)+17(1) \\ 13(3)+17(2)+26(1) \end{bmatrix}$$ Multiply columns by rows.

$$= \begin{bmatrix} 201 \\ 180 \\ 131 \\ 115 \\ 99 \end{bmatrix}$$ Simplify.

The labels for the product matrix are show below.

	Total Points
United States	201
Russia	180
China	131
Australia	115
Germany	99

Examine M is a 5×3 matrix and P is a 3×1 matrix, so their product should be a 5×1 matrix.

Example 4 Commutative Property

Find each product is $A = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$ **and** $B = \begin{bmatrix} 0 & 2 \\ -1 & 7 \end{bmatrix}$.

a. ***AB***

$$AB = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 \\ -1 & 7 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 1(0)+(-2)(-1) & 1(2)+(-2)(7) \\ 4(0)+(-3)(-1) & 4(2)+(-3)(7) \end{bmatrix}$$ Multiply columns by rows.

$$= \begin{bmatrix} 2 & -12 \\ 3 & -13 \end{bmatrix}$$ Simplify.

b. ***BA***

$$BA = \begin{bmatrix} 0 & 2 \\ -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 0(1)+2(4) & 0(-2)+2(-3) \\ -1(1)+7(4) & -1(-2)+7(-3) \end{bmatrix}$$ Multiply columns by rows.

$$= \begin{bmatrix} 8 & -6 \\ 27 & -19 \end{bmatrix}$$ Simplify.

Notice that $AB \neq BA$ in Example 4 since $\begin{bmatrix} 2 & -12 \\ 3 & -13 \end{bmatrix} \neq \begin{bmatrix} 8 & -6 \\ 27 & -19 \end{bmatrix}$.

Example 5 Distributive Property

Use $D = \begin{bmatrix} 0 & 3 \\ -2 & 4 \end{bmatrix}$, $E = \begin{bmatrix} -1 & 4 \\ -5 & 2 \end{bmatrix}$, **and** $F = \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}$.

a. ***ED + FD***

$$ED + FD = \begin{bmatrix} -1 & 4 \\ -5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1(0)+4(-2) & -1(3)+4(4) \\ -5(0)+2(-2) & -5(3)+2(4) \end{bmatrix} + \begin{bmatrix} -1(0)+(-3)(-2) & -1(3)+(-3)(4) \\ 2(0)+4(-2) & 2(3)+4(4) \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 13 \\ -4 & -7 \end{bmatrix} + \begin{bmatrix} 6 & -15 \\ -8 & 22 \end{bmatrix} \text{ or } \begin{bmatrix} -2 & -2 \\ -12 & 15 \end{bmatrix}$$

b. **(*E* + *F*)*D***

$$(E + F)D = \left(\begin{bmatrix} -1 & 4 \\ -5 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}\right) \cdot \begin{bmatrix} 0 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2(0)+1(-2) & -2(3)+1(4) \\ -3(0)+6(-2) & -3(3)+6(4) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 \\ -12 & 15 \end{bmatrix}$$