Key Concepts

Lesson 30

More on Axis of Symmetry and Vertices

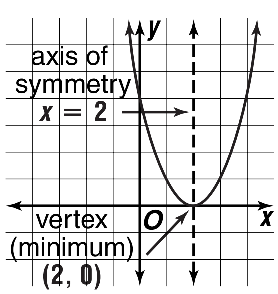
Objective Develop the formulas for the axis of symmetry and the vertex of a parabola, both geometrically and algebraically.

Note to the Teacher *In Lessons 11-1 and 11-1B, students studied how changing the coefficients in a quadratic function $y = ax^2 + bx + c$ affects the shape of its graph. They also learned that the axis of symmetry is the vertical line $x = -\frac{b}{2a}$ and that this formula can be used to find the coordinates of the vertex of a parabola. The goal of this lesson is to explain why this is true.*

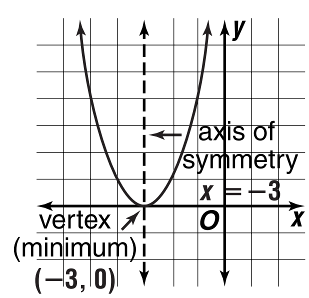
Graphs of $y = (x + d)^2$

Remind students how a parabola changes shape as the coefficients of $y = ax^2 + bx + c$ change. For example, the graph of $y = (x + d)^2$ has an axis of symmetry $x = -d$. This comes from the formula $x = -\frac{b}{2a}$ for the axis of symmetry, which was discussed in Lesson 11-1. Why is this true?

Notice that the value of $(x + d)^2$ is always positive or zero (since the square of a number can never be negative). That means that the smallest value of $(x + d)^2$ is zero, which occurs when $x + d = 0$, or $x = -d$. So the minimum point on the graph of $y = (x + d)^2$ occurs when $x = -d$ and $y = 0$. Since the axis of symmetry goes through the vertex (which in this case is a minimum point), it therefore occurs when $x = -d$.



$y = (x - 2)^2$
$d = -2$

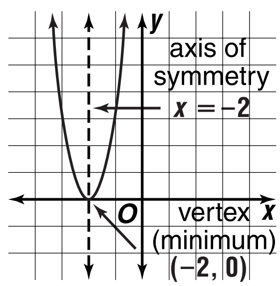


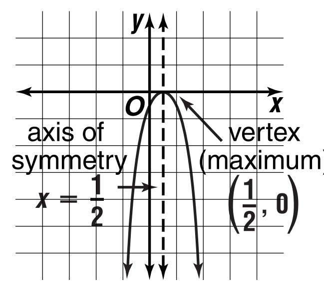
$y = (x + 3)^2$
$d = 3$

- Notice that when a is a positive number, then the function $y = a(x + d)^2$ is always greater than or equal to zero. The function is equal to zero when $a(x + d)^2 = 0$, which occurs when $(x + d) = 0$, or $x = -d$. The axis of symmetry for these functions is the vertical line through the minimum point (vertex), $x = -d$.
- When a is a negative number, a similar argument applies. Namely, since the value of $(x + d)^2$ is always greater than or equal to zero, then assuming a is a negative number, $y = a(x + d)^2$ is always less than or equal to zero. (Ask your students why.) This expression therefore has its maximum value at zero, which occurs when $a(x + d)^2 = 0$, or $x = -d$. Thus, the axis of symmetry, the vertical line through the maximum point (vertex), has the equation $x = -d$.

Graphs of $y = a(x + d)^2$

For any nonzero number a, the graph of $y = a(x + d)^2$ has a vertex at $(-d, 0)$, and the axis of symmetry is the vertical line $x = -d$. If $a > 0$, the vertex is a minimum point, and if $a < 0$, the vertex is a maximum point.





Graphs of $y = a(x + d)^2 + k$

In Lesson 11-1B, students learned that

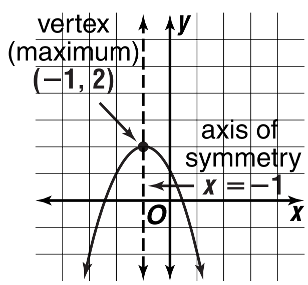
- when a positive number k is added to a quadratic function, $y = f(x) + k$, then the parabola shifts up k units, and
- when k is subtracted from a quadratic function, $y = f(x) - k$, then the parabola shifts down k units.

This is because if the graph of $y = f(x)$ has a minimum point (the parabola opens up), then the minimum value of $y = f(x) + k$ is k more than the minimum value of $y = f(x)$. Moreover, the minimum values of $y = f(x) + k$ and $y = f(x)$ occur at the same x value. So, if the graph of $y = f(x)$ has a minimum point at (x, y), then the graph of $y = f(x) + k$ has a minimum point at $(x, y + k)$. Since the x value of the minimum point is unchanged, the equation of the axis of symmetry is also unchanged.

In the same way, if the quadratic function $y = f(x)$ has a maximum point at (x, y) (the parabola opens down), then the maximum point of the graph of $y = f(x) + k$ is $(x, y + k)$.

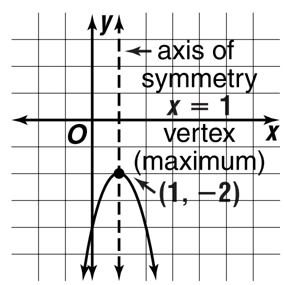
We can now apply this discussion to the functions of the form $f(x) = a(x + d)^2$ that we have just been studying. The graphs of quadratic functions of the form $y = f(x) + k$ or $y = a(x + d)^2 + k$ have the following properties, where a is nonzero, and d and k are any numbers.

1. The axis of symmetry is the line $x = -d$.
2. The vertex is at $(-d, k)$. If a is positive, the parabola opens up and the vertex is a minimum point. If a is negative, the parabola opens down and the vertex is a maximum point.



$y = -(x + 1)^2 + 2$

$a = -1, d = 1, k = 2$



$y = -2(x - 1)^2 - 2$

$a = -2, d = -1, k = -2$

Now notice that if we write $y = a(x + d)^2 + k$ in standard form $y = ax^2 + bx + c$, we can see how to get the formula $x = -\frac{b}{2a}$ for the axis of symmetry.

$$\begin{aligned} y &= a(x + d)^2 + k \\ &= a(x^2 + 2dx + d^2) + k \\ &= \underset{\substack{\uparrow \\ a}}{ax^2} + \underset{\substack{\uparrow \\ b}}{(2ad)}x + \underset{\substack{\uparrow \\ c}}{(ad^2 + k)} \end{aligned}$$

Write this calculation on the chalkboard. Comparing the coefficient of x, we have

$$\begin{aligned} 2ad &= b \\ d &= \frac{b}{2a}. \end{aligned}$$

Since the graph of $y = a(x + d)^2 + k$ has an axis of symmetry $x = -d$, this is equal to the equation $x = -\frac{b}{2a}$, which is the formula that was given in Lesson 11-1.

End this lesson by providing the following summary. Explain that any quadratic function $y = ax^2 + bx + c$ can be written in the form $y = a(x + d)^2 + k$ by a process called **completing the square**, which students will learn about in Chapter 12. Therefore, these justifications for the formulas for the axis of symmetry and the vertex work for any quadratic function.

Function	Effect on Graph	Axis of Symmetry	Vertex
$y = ax^2$	Graph widens ($a < 1$) or narrows ($a > 1$). Graphs opens up ($a > 0$) or down ($a < 0$).	$x = \frac{b}{2a} = \frac{0}{2a}$ or 0	$y = a(0)^2 = 0$ Vertex is at (0, 0).
$y = x^2 + c$	Graph moves up.	$x = -\frac{b}{2a} = -\frac{0}{2a}$ or 0	$y = (0)^2 + c = c$ Vertex is at (0, c).
$y = x^2 - c$	Graph moves down.		$y = (0)^2 - c = -c$ Vertex is at (0, $-c$).
$y = (x + d)^2$	Graph moves left.	$x = -\frac{b}{2a} = -\frac{2d}{2}$ or $-d$	$y = (-d + d)^2 = 0$ Vertex is at ($-d$, 0).
$y = (x - d)^2$	Graph moves right.	$x = -\frac{b}{2a} = \frac{2d}{2}$ or d	$y = (d - d)^2 = 0$ Vertex is at (d, 0).
$y = a(x + d)^2 + k$	Width and direction of graph depend on a.	$x = -d$	$y = a(-d + d)^2 + k$ $= 0 + k$ or k Vertex is at ($-d$, k).
$y = a(x - d)^2 + k$	Width and direction of graph depend on a.	$x = d$	$y = a[d + (-d)]^2 + k$ $= 0 + k$ or k Vertex is at (d, k).

End of Lesson