Key Concepts

Lesson 34

Growth and Decay

Objective Teach students to use exponential functions to solve problems involving exponential growth and decay.

Note to the Teacher *Begin by explaining to the class that many problems involving growth and decay can be described using exponential functions.*

Exponential Growth

When a population is increasing, the general equation describing it is referred to as exponential growth. It is given by the following general equation.

$$y = C(1 + r)^t$$

In this equation, y stands for the population at time t, C is the initial population (at time $t = 0$), and r is the rate at which the population is increasing per unit of time.

Example 1 **The population of Pleasantville is 100,000 in the year 2000. The population is increasing at a rate of 4% per year. What will the population be in the year 2015?**

Solution Use the general equation for exponential growth. Start measuring the population of Pleasantville in the year 2000, so in the year 2000, $t = 0$. The population at that time is 100,000, so $C = 100{,}000$. The rate of increase of the population is 4% per year. Convert 4% to a fraction, so $r = \frac{4}{100}$. The population at any time t is given by the following equation.

$$y = 100{,}000\left(1 + \frac{4}{100}\right)^t$$

$$y = 100{,}000\left(\frac{104}{100}\right)^t$$

$$y = 100{,}000\left(\frac{26}{25}\right)^t$$

In the year 2015, it will be 15 years since we started measuring the population of Pleasantville, and so the value of t is 15. Substituting in this equation, the population of Pleasantville in the year 2015 will be

$$y = 100{,}000\left(\frac{26}{25}\right)^{15}.$$

Use a calculator to approximate this number. The population will be approximately 180,094.

Exponential Decay

When a population is decreasing, the general equation describing it is referred to as *exponential decay*. It is given by the following general equation.

$$y = C(1 - r)^t$$

In this equation, r is the rate at which the population is decreasing. (Note that this is really the same equation as for exponential growth, except that the rate of change is now negative.) Do the following example from the Student Edition on the chalkboard.

Example 2 **(from p. 648 of the Student Edition) In 1980, there were 1.2 million elephants living in Africa. Because the natural grazing lands for the elephant are disappearing due to increased population and cultivation of the land, the number of elephants in Africa has decreased by about 6.8% per year. Write an equation to represent the population of elephants in Africa. In what year did the population of African elephants drop to less than half the number in 1980?**

Solution Use the general equation for exponential decay. Start measuring the population of elephants in 1980, so this is when $t = 0$. The initial population is 1.2 million, so $C = 1{,}200{,}000$. The rate of decrease is 6.8% per year, so $r = 0.068$. The population of African elephants at time t is therefore given by the following equation.

$$y = 1{,}200{,}000(1 - 0.068)^t$$
$$y = 1{,}200{,}000(0.932)^t$$

To find out when the population of elephants dropped to less than half the number in 1980 (to 600,000), solve for t.

$$600{,}000 = 1{,}200{,}000(0.932)^t$$
$$0.5 = (0.932)^t$$

Using a calculator, we find that $(0.932)^9$ is approximately 0.53, and $(0.932)^{10}$ is approximately 0.49. Since 0.5 is closer to 0.49 than to 0.53, then $t = 10$. Therefore, the year in which the population of elephants dropped to less than half the number in 1980 is $t = 10$ years after 1980, or 1990.

A special case of exponential growth is money, which compounds interest. In this case the growth equation becomes

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

In this formula, P is the initial amount of money in the investment (called the **principal**), A represents the amount of money in the investment after t years, r is the annual interest rate, and n is the number of times the interest is compounded each year.

Example 3 **In the year 2000, I put $100 in the bank, earning an interest rate of 5% per year, compounded every 3 months. If I don't deposit or withdraw any money from this account, when is the first year in which I will have at least $200 in the bank?**

Solution Use the growth equation for compound interest. In this case, start at time $t = 0$ in the year 2000. The initial amount (principal) is $P = 100$, the interest rate is 5%, so $r = 0.05$. The interest is compounded every 3 months, or 4 times per year. So $n = 4$. The amount of money in the bank at time t is therefore given by the following equation.

$$A = 100\left(1 + \frac{0.05}{4}\right)^{4t}$$

$$A = 100(1.0125)^{4t}$$

So, to find out when the amount in the bank is $200, solve for time t.

$$200 = 100(1.0125)^{4t}$$

$$2 = (1.0125)^{4t}$$

Using a calculator, we find that $(1.0125)^{55} \approx 1.98$ and $(1.0125)^{56} \approx 2.005$. Since 2 is closer to 2.005 than to 1.98, then $4t = 56$, or $t = 14$. So, after 14 years, there will be more than $200 in the account. That is, the account will double in the year 2014.