Key Concepts

Lesson 9-6

Theoretical and Experimental Probability

Objective Teach students the terminology used in probability theory, and how to make calculations pertaining to experiments where all outcomes are equally likely.

Note to the Teacher *Probability theory is a part of mathematics used to predict the likelihood of certain kinds of events. We can then make judgments and decisions based on these predictions. It is an extremely useful subject in many applications.*

Terminology

In order to talk about probability, we need to introduce some terminology. The study of probability involves conducting **probability experiments**, which are tests that have a set of definite **outcomes** (results) and which can be repeated many times. The goal of probability theory is to estimate how often we should expect a particular outcome to occur if we repeat the experiment many times.

Here are some examples of probability experiments.

- **Flipping a coin** There are two possible outcomes, heads and tails. This experiment can be repeated by flipping the coin again.
- **Drawing a card from a deck of playing cards** There are 52 possible outcomes, one for each card in the deck. This experiment can be repeated by replacing the card in the deck, shuffling the deck, and drawing a card again.
- **Rolling a die** There are six possible outcomes, since there are six faces on the die. The experiment can be repeated by rolling the die again.
- **Tossing two coins** In this case, there are four possible outcomes, which can be identified as (heads, heads), (heads, tails), (tails, heads), and (tails, tails). This experiment can be repeated by tossing the two coins again.

Key Idea	The set of all possible outcomes for a probability experiment is called the **sample space** for the experiment.

The following examples identify the sample spaces for three of the experiments listed on the previous page.

- When an experiment consists of flipping a coin once, the sample space is the set {heads, tails}. So, the sample space has two members.
- When an experiment consists of drawing one card from a standard deck of playing cards, the sample space consists of the 52 cards in the deck.
- When an experiment consists of rolling a die once, the sample space is the set {1, 2, 3, 4, 5, 6}. So, the sample space has six members.

Key Idea	An **event** is a specific outcome or type of outcome for a probability experiment.

Stress that an event does not have to be just the occurrence of a single outcome, like rolling a 6 on a die, but it can also be the occurrence of one of several related outcomes. The following examples point out this fact.

- When the experiment is rolling a die once, one possible event is *rolling an even number.* In this case, there are three different ways for the event to occur: rolling a 2, rolling a 4, and rolling a 6.
- When the experiment is drawing one card from a deck of cards, one possible event could be *drawing a red jack.* This event can occur in two ways: drawing the jack of hearts and drawing the jack of diamonds.

Note that events are often described verbally rather than by a list. So, for example, we could speak of the event that the role of a die gives an even number, or that a card drawn is a heart, rather than actually listing all the even numbers or all the cards that are hearts.

A **trial** is the act of carrying out an experiment one time. The goal of probability theory is to estimate what the results will be if we were to carry out a large number of trials of an experiment. Usually, we want to estimate how many times a particular event will occur.
We can best describe a situation using the ratio of the number of times an event occurs to the total number of trials done. This is stated formally below.

Key Idea	For an experiment, the **probability** of an event is the ratio $\frac{\text{number of times the event occurs}}{\text{number of trials}}$. Notice that in this fraction, the numerator can never be greater than the denominator. Therefore, the probability is *always* a number between 0 and 1, inclusive.

If we denote an event by the letter A, we will sometimes write $P(\mathrm{A})$ to mean the *probability of A*.

Note to the Teacher *We have not given a precise definition for probability, but rather a description of how to find a probability. The reason we have not given a precise definition is that the probability of an event is actually defined as the number which the ratio*

$$\frac{\textit{number of times the event occurs}}{\textit{number of trials}}$$

approaches as the number of trials becomes very large. This kind of limiting procedure is difficult to make precise at this stage, so we choose not to introduce it formally here. It is a good idea to talk about it with your class though, so that they understand that the probability is an estimate of the ratio when the number of trials is very large.

Finding Probabilities

Notice that if the probability of an event is 1, this means that the numerator in the fraction above is equal to the denominator. A probability of 1 means the event occurs *every* time a trial is done. Therefore we say that the event is **certain**. If an event can *never* occur, then its probability is 0 and we say the event is **impossible**.

Knowing the probability of an event is very useful in predicting how many times the event will occur when we perform a large number of trials. Here are some examples that can reinforce the idea of probability as an estimate.

Example 1 **An experiment involves drawing a card at random from a well-shuffled deck. We are told that the probability of drawing a 3 is $\frac{1}{13}$. Suppose we repeat this experiment 10,000 times. How many times should we expect to draw a 3?**

Solution Since the probability of drawing a 3 is $\frac{1}{13}$, our best estimate for the total number of times we will draw a 3 is $\frac{1}{13}$ times the total number of trials. In this case, the probability is

$$\frac{1}{13} \times 10{,}000,$$

which is approximately $0.077 \times 10{,}000$ or 770. So we can estimate that a 3 would be drawn about 770 times in 10,000 trials.

Experimental Versus Theoretical Probability

Determining probabilities can be done **experimentally** as well as theoretically. When we determine the probability of an event experimentally, we perform a large number of trials of an experiment and compute the ratio

$$\frac{\text{number of times the event occurs}}{\text{number of trials}}.$$

This ratio gives us a good estimate of how likely it is for the event to occur in future trials of the experiment.

Example 2 **There has been snow on the ground in Boston on December 25 for 210 of the last 300 years. Determine the probability of having snow on the ground in Boston on December 25 this year.**

Solution The probability is the ratio

$$\frac{\text{number of years with snow}}{\text{number of years}} = \frac{210}{300} \text{ or } 0.7.$$

So it is reasonable to estimate that there is a $\frac{7}{10}$ chance (also read "7 in 10 chance" or "70% chance") that there will be snow on the ground in Boston on December 25 this year.

Example 3 **A softball player's current batting average for this season is .320. Suppose she is batting in the game today. Find a good estimate for the likelihood that she will get a hit in this at-bat.**

Solution Each of the player's at-bats is regarded as a trial of the experiment, and the outcomes are *hit* or *not a hit*. Her batting average was obtained by dividing the total number of hits she has this season by the total number of at-bats, or the total number of trials. So her batting average is the probability of her getting a hit. The probability 0.320 can also be expressed as the fraction $\frac{320}{1,000}$ or $\frac{8}{25}$.

Have students estimate the likelihood of an event occurring by computing probabilities from various kinds of data. For instance, they could determine the probability that it rains on a given day in your town by using the number of days of rain over the last year. Students might also gather information about the students at their school to determine the probability that a student rides his/her bike to school, or the probability that a student rides the bus. This can be a group project, with each group presenting their findings to the rest of the class.

Computing Theoretical Probability

Conducting an experiment a sufficient number of times for empirical probability to be useful often requires an enormous amount of time. Usually the time requirement makes conducting an experiment impractical. When we have additional information about an experiment, we can sometimes evaluate the probability theoretically rather than experimentally. Here are a couple of examples.

Example 4 **The experiment is tossing a coin. What is the probability of the event {heads}?**

Solution If the coin is fair, then neither side of the coin is more likely to land up. So if we toss the coin many times (we carry out many trials), we can expect to obtain roughly as many heads as tails. This means that on average one of every two trials will land heads, so the probability of heads is $\frac{1}{2}$. (In the same way, the probability of tails is also $\frac{1}{2}$.)

Example 5 **The experiment is rolling a die.**

a. What is the probability of the event {6}?

b. What is the probability of the event {2, 4, 6}?

Solution We assume that the die is fair, meaning that each face will come up about the same number of times when we make a large number of rolls.

a. This means that on average a 6 will occur once in every six rolls, so we say that the probability of the event {6} is $\frac{1}{6}$.

b. Since the event {2, 4, 6} includes three of the six possible outcomes, the probability of this event is $\frac{3}{6}$ or $\frac{1}{2}$.

Note to the Teacher *This is a good place to have the class perform trials with coins and dice. Have them choose an event and then verify that as the number of trials increases, the ratio*

$$\frac{\textit{number of times the event occurs}}{\textit{number of trials}}$$

nears the theoretical probability for their event.

Probability When All Outcomes Occur Equally Often

In the previous two examples, we had additional information about the experiment being conducted. In both cases, the additional information was that each of the outcomes was expected to occur equally often when a large number of trials are performed. This information permitted us to compute probabilities precisely and easily.

Key Idea	When all outcomes in an experiment are equally likely, the probability that one particular outcome will occur is the ratio $\frac{1}{\text{number of possible outcomes}}$.

Why Is This True?

Suppose there are n possible outcomes and each outcome is equally likely. For example, when flipping a coin there are $n = 2$ possible outcomes (heads and tails) and each is equally likely. When rolling a die there are $n = 6$ possible outcomes, and each is equally likely.

If we do the experiment a large number of times, N, then we could divide the N experiments up into n parts, where each part would correspond to those experiments in which a particular outcome occurred. For example, we would divide the total number of coin flips into two parts, those corresponding to heads and those corresponding to tails. Since the events are all equally likely, we would expect each of the n parts of the N experiments to be the same number. In other words, the number of experiments in each part is $\frac{N}{n}$. So we can expect that the number of experiments in which a given outcome would occur is $\frac{N}{n}$. Therefore the probability of a given event is

$$\frac{\text{number of times the outcome occurs}}{\text{number of trials}} = \frac{\frac{N}{n}}{N} = \frac{1}{n}.$$

Example 6 **A single card is drawn from a well-shuffled deck of cards. What is the probability that the card is the king of spades?**

Solution There are 52 possible outcomes, one of which is the king of spades. All will occur equally often since the deck has been well-shuffled. So, the probability of drawing a particular card (in this case, the king of spades) is the fraction $\frac{1}{52}$.

Example 7 **In an experiment where two fair coins are tossed, what is the probability that both coins come up heads?**

Solution There are four possible outcomes: (heads, heads), (heads, tails), (tails, heads), and (tails, tails). Since each coin is fair, each outcome will occur equally often if we do many trials. So, the probability is

$$\frac{1}{\text{number of possible outcomes}} = \frac{1}{4}.$$

In a situation where all outcomes are equally likely, we can also determine the probability of an event, not just a particular outcome.

Key Idea	In an experiment where each outcome occurs equally often, the probability of an event is the ratio $\frac{\text{number of times the event occurs}}{\text{number of trials}}$.

Example 8 **Suppose we roll a fair die. What is the probability that the number we roll is divisible by 3?**

Solution Since the die is fair, each of the six possible outcomes is equally likely to occur. In this case, the event is the collection of all outcomes that are divisible by 3. There are two such outcomes, 3 and 6. So, the number of outcomes in the event is 2 and the number of possible outcomes is 6. Therefore, the probability that the number is divisible by 3 is $\frac{2}{6}$ or $\frac{1}{3}$.

Example 9 **Without looking, we draw one card from a well-shuffled deck. What is the probability that we draw a face card?**

Solution Since the deck is well-shuffled and we draw without looking, each card in the deck is equally likely to be drawn. There are 52 outcomes in all, one for each card. There are 12 face cards (4 jacks, 4 queens, and 4 kings), so the probability of drawing a face card is

$$\frac{\text{number of face cards}}{\text{number of cards in the deck}} = \frac{12}{52} \text{ or } \frac{3}{13}.$$

Example 10 **Two fair coins are tossed. What is the probability that exactly one of the coins lands heads up?**

Solution Since the coins are fair, all outcomes are equally likely. There are four outcomes in all. Two of them, (heads, tails) and (tails, heads), have exactly one head. So the probability of having exactly one head is

$$\frac{\text{number of outcomes with exactly one head}}{\text{number of trials}} = \frac{2}{4} \text{ or } \frac{1}{2}.$$

Why Does This Work?

The probability of the event can be obtained by adding the probabilities of each of the outcomes in the event. Since each of the outcomes is equally likely (meaning they have equal probabilities), this addition amounts to the repeated addition of the same value. The result is the same result we get by multiplying the probability of a particular outcome by the number of outcomes in the event:

$$(\text{number of outcomes in the event}) \times P(\text{an outcome}).$$

Since the probability of an outcome is

$$\frac{1}{\text{number of possible outcomes}},$$

this multiplication gives us the fraction

$$\frac{\text{number of outcomes in the event}}{\text{number of possible outcomes}}.$$

Note to the Teacher *Conclude the lesson by having students compare the probabilities obtained by the theoretical methods with the results of actually carrying out an experiment of their choice. The students could flip coins, draw cards, or draw marbles from a jar as their experiment, and compare their results from a large number of trials with the computed probabilities of their selected event.*