

# Recursion and Iteration

In a *recursive formula*, each succeeding term is formulated from one or more previous terms. A recursive formula for a sequence has two parts:

1. the value(s) of the first term(s), and
2. an equation that shows how to find each term from the term(s) before it.

## Example 1 Use a Recursive Formula

Find the first five terms of the sequence in which  $a_1 = 6$ ,  $a_2 = 10$ , and  $a_n = 2a_{n-2}$  for  $n \geq 3$ .

$$a_1 = 6$$

$$a_2 = 10$$

$$a_3 = 2a_1 = 2(6) = 12$$

$$a_4 = 2a_2 = 2(10) = 20$$

$$a_5 = 2a_3 = 2(12) = 24$$

The first five terms of the sequence are 6, 10, 12, 20, and 24.

## Exercises

Find the first five terms of each sequence described.

1.  $a_1 = 1$ ,  $a_2 = 1$ ,  $a_n = 2(a_{n-1} + a_{n-2})$ ,  $n \geq 3$

2.  $a_1 = 1$ ,  $a_n = \frac{1}{1 + a_{n-1}}$ ,  $n \geq 2$

3.  $a_1 = 3$ ,  $a_n = a_{n-1} + 2(n-2)$ ,  $n \geq 2$

4.  $a_1 = 5$ ,  $a_n = a_{n-1} + 2$ ,  $n \geq 2$

5.  $a_1 = 0.5$ ,  $a_n = a_{n-1} + 2n$ ,  $n \geq 2$

6.  $a_1 = 100$ ,  $a_n = \frac{a_{n-1}}{n}$ ,  $n \geq 2$

Write a recursive formula for each sequence.

7.  $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \dots$

8.  $1, -1, 2, -3, 5, -8, \dots$

## Recursion and Iteration (continued)

Combining composition of functions with the concept of recursion leads to the process of *iteration*. Iteration is the process of composing a function with itself repeatedly.

### Example 2 Find Iterates of a Function

Find the first three iterates of  $f(x) = 4x - 5$  for an initial value of  $x_0 = 2$ .

To find the first iterate, find the value of the function for  $x_0 = 2$ .

$$\begin{aligned}x_1 &= f(x_0) && \text{Iterate the function.} \\ &= f(2) && x_0 = 2 \\ &= 4(2) - 5 \text{ or } 3 && \text{Simplify.}\end{aligned}$$

To find the second iterate, find the value of the function for  $x_1 = 3$ .

$$\begin{aligned}x_2 &= f(x_1) && \text{Iterate the function.} \\ &= f(3) && x_1 = 3 \\ &= 4(3) - 5 \text{ or } 7 && \text{Simplify.}\end{aligned}$$

To find the third iterate, find the value of the function for  $x_2 = 7$ .

$$\begin{aligned}x_3 &= f(x_2) && \text{Iterate the function.} \\ &= f(7) && x_2 = 7 \\ &= 4(7) - 5 \text{ or } 23 && \text{Simplify.}\end{aligned}$$

The first three iterates are 3, 7, and 23.

### Exercises

Find the first three iterates of each function for the given initial value.

9.  $f(x) = x - 1; x_0 = 4$

10.  $f(x) = x^2 - 3x; x_0 = 1$

11.  $f(x) = x^2 + 2x + 1; x_0 = -2$

12.  $f(x) = 4x - 6; x_0 = -5$

13.  $f(x) = 6x - 2; x_0 = 3$

14.  $f(x) = 100 - 4x; x_0 = -5$

15.  $f(x) = 3x - 1; x_0 = 47$

16.  $f(x) = x^3 - 5x^2; x_0 = 1$

17.  $f(x) = 10x - 25; x_0 = 2$

18.  $f(x) = 4x^2 - 9; x_0 = -1$

19.  $f(x) = 2x^2 + 5; x_0 = -4$

20.  $f(x) = \frac{x-1}{x+2}; x_0 = 1$

21.  $f(x) = \frac{1}{2}(x + 11); x_0 = 3$

22.  $f(x) = \frac{3}{x}; x_0 = 9$

23.  $f(x) = x - 4x^2; x_0 = 1$

24.  $f(x) = x + \frac{1}{x}; x_0 = 2$

25.  $f(x) = x^3 - 5x^2 + 8x - 10;$   
 $x_0 = 1$

26.  $f(x) = x^3 - x^2; x_0 = -2$

## Recursion and Iteration

### Answers

1. 1, 1, 4, 10, 28

2.  $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}$

3. 3, 3, 5, 9, 15

4. 5, 7, 9, 11, 13

5. 0.5, 4.5, 10.5, 18.5, 28.5

6. 100, 50,  $\frac{50}{3}, \frac{25}{6}, \frac{5}{6}$

7.  $a_1 = 1, a_n = \frac{1}{n} a_{n-1}, n \geq 2$

8.  $a_1 = 1, a_2 = -1, a_n = a_{n-2} - a_{n-1}, n \geq 3$

9. 3, 2, 1

10. -2, 10, 70

11. 1, 4, 25

12. -26, -110, -446

13. 16, 94, 562

14. 120; -380; 1,620

15. 140; 419; 1,256

16. -4; -144; -3,089,664

17. -5, -75, -775

18. -5; 91; 33,115

19. 37; 2,743; 15,048,103

20.  $0, -\frac{1}{2}, -1$

21. 7, 9, 10

22.  $\frac{1}{3}, 9, \frac{1}{3}$

23. -3; -39; -6,123

24.  $\frac{5}{2}, \frac{29}{10}, \frac{941}{290}$

25. -6; -454; -94,610,886

26. -12; -1,872; -6,563,711,232