Using Your North Carolina StudyText

*North Carolina StudyText, Math BC, Volume 2,* is a practice workbook designed to help you master the North Carolina Essential Standards for High School Math BC. By mastering the mathematics standards, you will be prepared to do well on your end-of-course (EOC) test. This StudyText is divided into two sections.

**Chapter Resources**

- Each chapter contains four pages for each key lesson in your *North Carolina Algebra 2* Student Edition. Your teacher may ask you to complete one or more of these worksheets as an assignment.

**Mastering the EOC**

This section of StudyText is composed of three parts. Each part can help you study for your EOC test.

- The **Diagnostic Test** can help you determine which standards you might need to review before taking the EOC test. Each question lists the standard that it is assessing. Your teacher may assign review pages based on the questions that you did not answer correctly.

- **Practice by Standard** gives you more practice problems to help you become a better test-taker. The problems are organized by the North Carolina High School Math BC Essential Standards. You can also use these pages as a general review before you take the EOC test.

- The **Practice Test** can be used to simulate what an EOC test might be like so that you will be better prepared to take it in the spring.
## Contents in Brief

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### Mastering the EOC, Algebra 2 / Math BC, Volume 2

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Expressions and Formulas

Order of Operations

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Example 1 Evaluate \([18 - (6 + 4)] \div 2\).

\[
[18 - (6 + 4)] \div 2 = [18 - 10] \div 2
\]

\[
= 8 \div 2
\]

\[
= 4
\]

Example 2 Evaluate \(3x^2 + x(y - 5)\) if \(x = 3\) and \(y = 0.5\).

Replace each variable with the given value.

\[
3x^2 + x(y - 5) = 3 \cdot (3)^2 + 3(0.5 - 5)
\]

\[
= 3 \cdot (9) + 3(-4.5)
\]

\[
= 27 - 13.5
\]

\[
= 13.5
\]

Exercises

Evaluate each expression.

1. \(14 + (6 \div 2)\)
2. \(11 - (3 + 2)^2\)
3. \(2 + (4 - 2)^3 - 6\)
4. \(9(3^2 + 6)\)
5. \((5 + 2^3)^2 - 5^2\)
6. \(5^2 + \frac{1}{4} + 18 \div 2\)
7. \(\frac{16 + 2^3 \div 4}{1 - 2^2}\)
8. \((7 - 3^2)^2 + 6^2\)
9. \(20 \div 2^2 + 6\)
10. \(12 + 6 \div 3 - 2(4)\)
11. \(14 \div (8 - 20 \div 2)\)
12. \(6(7) + 4 \div 4 - 5\)
13. \(8(4^2 \div 8 - 32)\)
14. \(\frac{6 + 4 \div 2}{4 \div 6 - 1}\)

Evaluate each expression if \(a = 8.2\), \(b = -3\), \(c = 4\), and \(d = -\frac{1}{2}\).

16. \(\frac{ab}{d}\)
17. \(5(6c - 8b + 10d)\)
18. \(\frac{c^2 - 1}{b - d}\)
19. \(ac - bd\)
20. \((b - c)^2 + 4a\)
21. \(\frac{a}{d} + 6b - 5c\)
22. \(3\left(\frac{c}{d}\right) - b\)
23. \(cd + \frac{b}{d}\)
24. \(d(a + c)\)
25. \(a + b \div c\)
26. \(b - c + 4 \div d\)
27. \(\frac{a}{b + c} - d\)
Expressions and Formulas

**Formulas** A formula is a mathematical sentence that expresses the relationship between certain quantities. If you know the value of every variable in the formula except one, you can use substitution and the order of operations to find the value of the remaining variable.

**Example** The formula for the number of reams of paper needed to print $n$ copies of a booklet that is $p$ pages long is $r = \frac{np}{500}$, where $r$ is the number of reams needed. How many reams of paper must you buy to print 172 copies of a 25-page booklet?

$$r = \frac{np}{500}$$  
Formula for paper needed

$$= \frac{(172)(25)}{500}$$  
$n = 172$ and $p = 25$

$$= \frac{4300}{500}$$  
Evaluate $(172)(25)$.

$$= 8.6$$  
Divide.

You cannot buy 8.6 reams of paper. You will need to buy 9 reams to print 172 copies.

**Exercises**

1. For a science experiment, Sarah counts the number of breaths needed for her to blow up a beach ball. She will then find the volume of the beach ball in cubic centimeters and divide by the number of breaths to find the average volume of air per breath.

   **a.** Her beach ball has a radius of 9 inches. First she converts the radius to centimeters using the formula $C = 2.54I$, where $C$ is a length in centimeters and $I$ is the same length in inches. How many centimeters are there in 9 inches?

   **b.** The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$, where $V$ is the volume of the sphere and $r$ is its radius. What is the volume of the beach ball in cubic centimeters? (Use 3.14 for $\pi$.)

   **c.** Sarah takes 40 breaths to blow up the beach ball. What is the average volume of air per breath?

2. A person's basal metabolic rate (or BMR) is the number of Calories needed to support his or her bodily functions for one day. The BMR of an 80-year-old man is given by the formula $\text{BMR} = 12w - (0.02)(6)12w$, where $w$ is the man's weight in pounds. What is the BMR of an 80-year-old man who weighs 170 pounds?
Evaluate each expression.

1. \(3(4 - 7) - 11\)
2. \(4(12 - 4^2)\)
3. \(1 + 2 - 3(4) \div 2\)
4. \(12 - [20 - 2(6^2 \div 3 \times 2^3)]\)
5. \(20 \div (5 - 3) + 5^2(3)\)
6. \((-2)^3 - (3)(8) + (5)(10)\)
7. \(18 - [5 - [34 - (17 - 11)]]\)
8. \([4(5 - 3) - 2(4 - 8)] \div 16\)
9. \(\frac{1}{2}[6 - 4^2]\)
10. \(\frac{1}{4}[-5 + 5(-3)]\)
11. \(-\frac{8(13 - 37)}{6}\)
12. \(\frac{(-8)^2}{5 - 9} - (-1)^2 + 4(-9)\)

Evaluate each expression if \(a = \frac{3}{4}, b = -8, c = -2, d = 3,\) and \(g = \frac{1}{3}\).

13. \(ab^2 - d\)
14. \((c + d)b\)
15. \(\frac{ab}{c} + d^2\)
16. \(\frac{d(b - c)}{ac}\)
17. \((b - dg)g^2\)
18. \(ac^3 - b^2dg\)
19. \(-b[a + (c - d)^2]\)
20. \(\frac{ac^4}{d} - \frac{c}{g^2}\)
21. \(9bc - \frac{1}{g}\)
22. \(2ab^2 - (d^3 - c)\)

23. TEMPERATURE The formula \(F = \frac{9}{5}C + 32\) gives the temperature in degrees Fahrenheit for a given temperature in degrees Celsius. What is the temperature in degrees Fahrenheit when the temperature is \(-40\) degrees Celsius?

24. PHYSICS The formula \(h = 120t - 16t^2\) gives the height \(h\) in feet of an object \(t\) seconds after it is shot upward from Earth’s surface with an initial velocity of 120 feet per second. What will the height of the object be after 6 seconds?

25. AGRICULTURE Faith owns an organic apple orchard. From her experience the last few seasons, she has developed the formula \(P = 20x - 0.01x^2 - 240\) to predict her profit \(P\) in dollars this season if her trees produce \(x\) bushels of apples. What is Faith’s predicted profit this season if her orchard produces 300 bushels of apples?
1-1 Word Problem Practice

Expressions and Formulas

1. ARRANGEMENTS The chairs in an auditorium are arranged into two rectangles. Both rectangles are 10 rows deep. One rectangle has 6 chairs per row and the other has 12 chairs per row. Write an expression for the total number of chairs in the auditorium.

2. GEOMETRY The formula for the area of a ring-shaped object is given by \( A = \pi(R^2 - r^2) \), where \( R \) is the radius of the outer circle and \( r \) is the radius of the inner circle. If \( R = 10 \) inches and \( r = 5 \) inches, what is the area rounded to the nearest square inch?

3. GUESS AND CHECK Amanda received a worksheet from her teacher. Unfortunately, one of the operations in an equation was covered by a blot. What operation is hidden by the blot?

   \[ 10 + 3(4 \times 6) = 4 \]

4. GAS MILEAGE Rick has \( d \) dollars. The formula for the number of gallons of gasoline that Rick can buy with \( d \) dollars is given by \( g = \frac{d}{3} \). The formula for the number of miles that Rick can drive on \( g \) gallons of gasoline is given by \( m = 21g \). How many miles can Rick drive on $8 worth of gasoline?

5. COOKING A steak has thickness \( w \) inches. Let \( T \) be the time it takes to broil the steak. It takes 12 minutes to broil a one-inch-thick steak. For every additional inch of thickness, the steak should be broiled for 5 more minutes.

   a. Write a formula for \( T \) in terms of \( w \).

   b. Use your formula to compute the number of minutes it would take to broil a 2-inch-thick steak.
Relations and Functions

A relation can be represented as a set of ordered pairs or as an equation; the relation is then the set of all ordered pairs \((x, y)\) that make the equation true. A function is a relation in which each element of the domain is paired with exactly one element of the range.

<table>
<thead>
<tr>
<th>One-to-One Function</th>
<th>Each element of the domain pairs to exactly one unique element of the range.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Onto Function</td>
<td>Each element of the range also corresponds to an element of the domain.</td>
</tr>
<tr>
<td>Both One-to-One and Onto</td>
<td>Each element of the domain is paired to exactly one element of the range and each element of the range.</td>
</tr>
</tbody>
</table>

**Example**

State the domain and range of the relation.

Does the relation represent a function?

The domain and range are both all real numbers. Each element of the domain corresponds with exactly one element of the range, so it is a function.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-5</td>
</tr>
<tr>
<td>0</td>
<td>-3</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

**Exercises**

State the domain and range of each relation. Then determine whether each relation is a function. If it is a function, determine if it is one-to-one, onto, both, or neither.

1. \(((0.5, 3), (0.4, 2), (3.1, 1), (0.4, 0))\)
2. \((-5, 2), (4, -2), (3, -11), (-7, 2))\)
3. \(((0.5, -3), (0.1, 12), (6, 8))\)
4. \((-15, 12), (-14, 11), (-13, 10), (-12, 12))\)
Equations of Relations and Functions Equations that represent functions are often written in functional notation. For example, \( y = 10 - 8x \) can be written as \( f(x) = 10 - 8x \). This notation emphasizes the fact that the values of \( y \), the dependent variable, depend on the values of \( x \), the independent variable.

To evaluate a function, or find a functional value, means to substitute a given value in the domain into the equation to find the corresponding element in the range.

**Example** Given \( f(x) = x^2 + 2x \), find each value.

a. \( f(3) \)

\[
\begin{align*}
  f(x) &= x^2 + 2x & \text{Original function} \\
  f(3) &= 3^2 + 2(3) & \text{Substitute.} \\
  &= 15 & \text{Simplify.}
\end{align*}
\]

b. \( f(5a) \)

\[
\begin{align*}
  f(x) &= x^2 + 2x & \text{Original function} \\
  f(5a) &= (5a)^2 + 2(5a) & \text{Substitute.} \\
  &= 25a^2 + 10a & \text{Simplify.}
\end{align*}
\]

**Exercises**

Graph each relation or equation and determine the domain and range. Determine whether the relation is a function, is one-to-one, onto, both, or neither. Then state whether it is discrete or continuous.

1. \( y = 3 \)  
2. \( y = x^2 - 1 \)  
3. \( y = 3x + 2 \)  

Find each value if \( f(x) = -2x + 4 \).

4. \( f(12) \)  
5. \( f(6) \)  
6. \( f(2b) \)

Find each value if \( g(x) = x^3 - x \).

7. \( g(5) \)  
8. \( g(-2) \)  
9. \( g(7c) \)
Practice

Relations and Functions

State the domain and range of each relation. Then determine whether each relation is a function. If it is a function, determine if it is one-to-one, onto, both, or neither.

1. Domain | Range
   2 | 21 25 30
   8

2. Domain | Range
   5 | 105 110
   10
   15

Graph each equation and determine the domain and range. Determine whether the relation is a function, is one-to-one, onto, both, or neither. Then state whether it is discrete or continuous.

3. \( \begin{array}{c|c} \hline x & y \\ \hline -3 & 0 \\ -1 & -1 \\ 0 & 0 \\ 2 & -2 \\ 3 & 4 \\ \hline \end{array} \)

4. \( \begin{array}{c|c} \hline x & y \\ \hline -2 & -1 \\ -2 & 1 \\ -1 & 0 \\ 1 & 0 \\ 2 & 1 \\ \hline \end{array} \)

Find each value if \( f(x) = \frac{5}{x + 2} \) and \( g(x) = -2x + 3. \)

7. \( f(3) \)
8. \( f(-4) \)
9. \( g\left(\frac{1}{2}\right) \)
10. \( f(-2) \)
11. \( g(-6) \)
12. \( f(m - 2) \)

13. MUSIC The ordered pairs (1, 16), (2, 16), (3, 32), (4, 32), and (5, 48) represent the cost of buying various numbers of CDs through a music club. Identify the domain and range of the relation. Is the relation discrete or continuous? Is the relation a function?

14. COMPUTING If a computer can do one calculation in 0.00000000015 second, then the function \( T(n) = 0.00000000015n \) gives the time required for the computer to do \( n \) calculations. How long would it take the computer to do 5 billion calculations?
1. **PLANETS** The table below gives the mean distance from the Sun and orbital period of the eight major planets in our Solar System. Think of the mean distance as the domain and the orbital period as the range of a relation. Is this relation a function? Explain.

<table>
<thead>
<tr>
<th>Planet</th>
<th>Mean Distance from Sun (AU)</th>
<th>Orbital Period (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>0.387</td>
<td>0.241</td>
</tr>
<tr>
<td>Venus</td>
<td>0.723</td>
<td>0.615</td>
</tr>
<tr>
<td>Earth</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Mars</td>
<td>1.524</td>
<td>1.881</td>
</tr>
<tr>
<td>Jupiter</td>
<td>5.204</td>
<td>11.75</td>
</tr>
<tr>
<td>Saturn</td>
<td>9.582</td>
<td>29.5</td>
</tr>
<tr>
<td>Uranus</td>
<td>19.201</td>
<td>84</td>
</tr>
<tr>
<td>Neptune</td>
<td>30.047</td>
<td>165</td>
</tr>
</tbody>
</table>

2. **PROBABILITY** Martha rolls a number cube several times and makes the frequency graph shown. Write a relation to represent this data.

3. **SCHOOL** The number of students $N$ in Vassia’s school is given by $N = 120 + 30G$, where $G$ is the grade level. Is 285 in the range of this function?

4. **FLOWERS** Anthony decides to decorate a ballroom with $r = 3n + 20$ roses, where $n$ is the number of dancers. It occurs to Anthony that the dancers always come in pairs. That is, $n = 2p$, where $p$ is the number of pairs. What is $r$ as a function of $p$?

5. **SALES** Cool Athletics introduced the new Power Sneaker in one of their stores. The table shows the sales for the first 6 weeks.

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pairs Sold</td>
<td>8</td>
<td>10</td>
<td>15</td>
<td>22</td>
<td>31</td>
<td>44</td>
</tr>
</tbody>
</table>

a. Graph the data.

b. Identify the domain and range.

c. Is the relation a function? Explain.
Linear Relations and Functions

A linear equation has no operations other than addition, subtraction, and multiplication of a variable by a constant. The variables may not be multiplied together or appear in a denominator. A linear equation does not contain variables with exponents other than 1. The graph of a linear equation is always a line.

A linear function is a function with ordered pairs that satisfy a linear equation. Any linear function can be written in the form $f(x) = mx + b$, where $m$ and $b$ are real numbers.

If an equation is linear, you need only two points that satisfy the equation in order to graph the equation. One way is to find the $x$-intercept and the $y$-intercept and connect these two points with a line.

**Example 1**  
Is $f(x) = 0.2 - \frac{x}{5}$ a linear function? Explain.

Yes; it is a linear function because it can be written in the form $f(x) = -\frac{1}{5}x + 0.2$.

**Example 2**  
Is $2x + xy - 3y = 0$ a linear function? Explain.

No; it is not a linear function because the variables $x$ and $y$ are multiplied together in the middle term.

**Exercises**

State whether each function is a linear function. Write **yes** or **no**. Explain.

1. $6y - x = 7$
2. $9x = \frac{18}{y}$
3. $f(x) = 2 - \frac{x}{11}$
4. $2y - \frac{x}{6} - 4 = 0$
5. $1.6x - 2.4y = 4$
6. $0.2x = 100 - \frac{0.4}{y}$
7. $f(x) = 4 - x^3$
8. $f(x) = \frac{4}{x}$
9. $2yx - 3y + 2x = 0$
**2-2 Study Guide (continued)**

**Linear Relations and Functions**

**Standard Form** The standard form of a linear equation is \( Ax + By = C \), where \( A, B, \) and \( C \) are integers whose greatest common factor is 1.

**Example 1** Write each equation in standard form. Identify \( A, B, \) and \( C \).

<table>
<thead>
<tr>
<th>Equation</th>
<th>Standard Form</th>
<th>Identify A, B, C</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = 8x - 5 )</td>
<td>(-8x + y = -5)</td>
<td>( A = 8, B = -1, C = 5 )</td>
</tr>
<tr>
<td>( 14x = -7y + 21 )</td>
<td>( 14x + 7y = 21 )</td>
<td>( A = 2, B = 1, C = 3 )</td>
</tr>
</tbody>
</table>

**Example 2** Find the \( x \)-intercept and the \( y \)-intercept of the graph of \( 4x - 5y = 20 \). Then graph the equation.

The \( x \)-intercept is the value of \( x \) when \( y = 0 \).

\[
4x - 5y = 20 \quad \text{Original equation} \\
4x - 5(0) = 20 \quad \text{Substitute 0 for } y \\
x = 5 \quad \text{Simplify.}
\]

So the \( x \)-intercept is 5. Similarly, the \( y \)-intercept is \(-4\).

**Exercises**

Write each equation in standard form. Identify \( A, B, \) and \( C \).

1. \( 2x = 4y - 1 \)
2. \( 5y = 2x + 3 \)
3. \( 3x = -5y + 2 \)
4. \( 18y = 24x - 9 \)
5. \( \frac{3}{4}y = \frac{2}{3}x + 5 \)
6. \( 6y - 8x + 10 = 0 \)
7. \( 0.4x + 3y = 10 \)
8. \( x = 4y - 7 \)
9. \( 2y = 3x + 6 \)

Find the \( x \)-intercept and the \( y \)-intercept of the graph of each equation. Then graph the equation using the intercepts.

10. \( 2x + 7y = 14 \)
11. \( 5y - x = 10 \)
12. \( 2.5x - 5y + 7.5 = 0 \)
2-2 Practice

Linear Relations and Functions

State whether each function is a linear function. Write yes or no. Explain.

1. \( h(x) = 23 \)
   2. \( y = \frac{2}{3}x \)

3. \( y = \frac{5}{x} \)
   4. \( 9 - 5xy = 2 \)

Write each equation in standard form. Identify \( A \), \( B \), and \( C \).

5. \( y = 7x - 5 \)
   6. \( y = \frac{3}{8}x + 5 \)

7. \( 3y - 5 = 0 \)
   8. \( x = \frac{2}{7}y + \frac{3}{4} \)

Find the \( x \)-intercept and the \( y \)-intercept of the graph of each equation. Then graph the equation using the intercepts.

9. \( y = 2x + 4 \)
10. \( 2x + 7y = 14 \)

11. \( y = -2x - 4 \)
12. \( 6x + 2y = 6 \)

13. MEASURE The equation \( y = 2.54x \) gives the length \( y \) in centimeters corresponding to a length \( x \) in inches. What is the length in centimeters of a 1-foot ruler?

14. LONG DISTANCE For Meg’s long-distance calling plan, the monthly cost \( C \) in dollars is given by the linear function \( C(t) = 6 + 0.05t \), where \( t \) is the number of minutes talked.
   a. What is the total cost of talking 8 hours? of talking 20 hours?
   b. What is the effective cost per minute (the total cost divided by the number of minutes talked) of talking 8 hours? of talking 20 hours?
1. WORK RATE  The linear equation \( n = 10t \) describes \( n \), the number of origami boxes that Holly can fold in \( t \) hours. How many boxes can Holly fold in 3 hours?

2. BASKETBALL  Tony tossed a basketball. Below is a graph showing the height of the basketball as a function of time. Is this the graph of a linear function? Explain.

3. PROFIT  Paul charges people $25 to test the air quality in their homes. The device he uses to test air quality cost him $500. Write an equation that describes Paul’s net profit as a function of the number of clients he gets. How many clients does he need to break even?

4. RAMP  A ramp is described by the equation \( 5x + 7y = 35 \). What is the area of the shaded region?

5. SWIMMING POOL  A swimming pool is shaped as shown below. The total perimeter is 500 feet.

   a. Write an equation that relates \( x \) and \( y \).

   b. Write the linear equation from part a in standard form.

   c. Graph the equation.

   d. Olympic swimming pools are 164 feet long. If this pool is an olympic pool, what is the value of \( y \)?
Scatter Plots and Prediction Equations

A set of data points graphed as ordered pairs in a coordinate plane is called a **scatter plot**. A scatter plot can be used to determine if there is a relationship among the data. A **line of fit** is a line that closely approximates a set of data graphed in a scatter plot. The equation of a line of fit is called a **prediction equation** because it can be used to predict values not given in the data set.

**Example**

**STORAGE COSTS** According to a certain prediction equation, the cost of 200 square feet of storage space is $60. The cost of 325 square feet of storage space is $160.

a. Find the slope of the prediction equation. What does it represent?

Since the cost depends upon the square footage, let \( x \) represent the amount of storage space in square feet and \( y \) represent the cost in dollars. The slope can be found using the formula

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

So,

\[
m = \frac{160 - 60}{325 - 200} = \frac{100}{125} = 0.8
\]

The slope of the prediction equation is 0.8. This means that the price of storage increases 80¢ for each one-square-foot increase in storage space.

b. Find a prediction equation.

Using the slope and one of the points on the line, you can use the point-slope form to find a prediction equation.

\[
y - y_1 = m(x - x_1)
\]

Point-slope form

\[
y - 60 = 0.8(x - 200)
\]

\((x, y) = (200, 60), m = 0.8\)

\[
y - 60 = 0.8x - 160
\]

Distributive Property

\[
y = 0.8x - 100
\]

Add 60 to both sides.

A prediction equation is \( y = 0.8x - 100 \).

**Exercises**

1. **SALARIES** The table below shows the years of experience for eight technicians at Lewis Techomatic and the hourly rate of pay each technician earns.

<table>
<thead>
<tr>
<th>Experience (years)</th>
<th>9</th>
<th>4</th>
<th>3</th>
<th>1</th>
<th>10</th>
<th>6</th>
<th>12</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hourly Rate of Pay</td>
<td>$17</td>
<td>$10</td>
<td>$10</td>
<td>$7</td>
<td>$19</td>
<td>$12</td>
<td>$20</td>
<td>$15</td>
</tr>
</tbody>
</table>

a. Draw a scatter plot to show how years of experience are related to hourly rate of pay. Draw a line of fit and describe the correlation.

b. Write a prediction equation to show how years of experience \( x \) are related to hourly rate of pay \( y \).

c. Use the function to predict the hourly rate of pay for 15 years of experience.
**Scatter Plots and Lines of Regression**

### Lines of Regression
Another method for writing a line of fit is to use a line of regression. A **regression line** is determined through complex calculations to ensure that the distance of all the data points to the line of fit are at the minimum.

#### Example
**WORLD POPULATION** The following table gives the United Nations estimates of the world population (in billions) every five years from 1980-2005. Find the equation and graph the line of regression. Then predict the population in 2010.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population (billions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980</td>
<td>4.451</td>
</tr>
<tr>
<td>1985</td>
<td>4.855</td>
</tr>
<tr>
<td>1990</td>
<td>5.295</td>
</tr>
<tr>
<td>1995</td>
<td>5.719</td>
</tr>
<tr>
<td>2000</td>
<td>6.124</td>
</tr>
<tr>
<td>2005</td>
<td>6.515</td>
</tr>
<tr>
<td>2010</td>
<td>?</td>
</tr>
</tbody>
</table>

**Source:** UN 2006 Revisions Population database

#### Step 1
Use your calculator to make a scatter plot.

#### Step 2
Find the equation of the line of regression. The equation is about \( y = 0.083x - 160.180 \).

#### Step 3
Graph the regression equation.

#### Step 4
Predict using the function.
In 2010 the population will be approximately 6.948 billion.

### Exercise

1. The table below shows the number of women who served in the United States Congress during the years 1995–2006. Find an equation for and graph a line of regression. Then use the function to predict the number of women in Congress in the 112th Congressional Session.

<table>
<thead>
<tr>
<th>Congressional Session</th>
<th>Number of Women</th>
</tr>
</thead>
<tbody>
<tr>
<td>104</td>
<td>59</td>
</tr>
<tr>
<td>105</td>
<td>65</td>
</tr>
<tr>
<td>106</td>
<td>67</td>
</tr>
<tr>
<td>107</td>
<td>75</td>
</tr>
<tr>
<td>108</td>
<td>77</td>
</tr>
<tr>
<td>109</td>
<td>83</td>
</tr>
</tbody>
</table>

**Source:** U. S. Senate
For Exercises 1 and 2, complete parts a–c.

a. Make a scatter plot and a line of fit, and describe the correlation.
b. Use two ordered pairs to write a prediction equation.
c. Use your prediction equation to predict the missing value.

1. **FUEL ECONOMY** The table gives the weights in tons and estimates the fuel economy in miles per gallon for several cars.

<table>
<thead>
<tr>
<th>Weight (tons)</th>
<th>1.3</th>
<th>1.4</th>
<th>1.5</th>
<th>1.8</th>
<th>2.0</th>
<th>2.1</th>
<th>2.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Miles per Gallon</td>
<td>29</td>
<td>24</td>
<td>23</td>
<td>21</td>
<td>?</td>
<td>17</td>
<td>15</td>
</tr>
</tbody>
</table>

![Fuel Economy Versus Weight Graph](image)

2. **ALTITUDE** As Anchara drives into the mountains, her car thermometer registers the temperatures (°F) shown in the table at the given altitudes (feet).

<table>
<thead>
<tr>
<th>Altitude (ft)</th>
<th>7500</th>
<th>8200</th>
<th>8600</th>
<th>9200</th>
<th>9700</th>
<th>10,400</th>
<th>12,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature (°F)</td>
<td>61</td>
<td>58</td>
<td>56</td>
<td>53</td>
<td>50</td>
<td>46</td>
<td>?</td>
</tr>
</tbody>
</table>

![Temperature Versus Altitude Graph](image)

3. **HEALTH** Alton has a treadmill that uses the time on the treadmill to estimate the number of Calories he burns during a workout. The table gives workout times and Calories burned for several workouts. Find an equation for and graph a line of regression. Then use the function to predict the number of Calories burned in a 60-minute workout.

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>18</th>
<th>24</th>
<th>30</th>
<th>40</th>
<th>42</th>
<th>48</th>
<th>52</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calories Burned</td>
<td>260</td>
<td>280</td>
<td>320</td>
<td>380</td>
<td>400</td>
<td>440</td>
<td>475</td>
<td>?</td>
</tr>
</tbody>
</table>

![Burning Calories Graph](image)
# Scatter Plots and Lines of Regression

## 1. AIRCRAFT
The table shows the maximum speed and altitude of different aircraft. Draw a scatter plot of this data.

<table>
<thead>
<tr>
<th>Max. Speed (knots)</th>
<th>121</th>
<th>123</th>
<th>137</th>
<th>173</th>
<th>153</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. Altitude (1000 feet)</td>
<td>14.2</td>
<td>17.0</td>
<td>15.3</td>
<td>20.7</td>
<td>16.0</td>
</tr>
</tbody>
</table>

*Source: RisingUp Aviation*

## 2. TESTING
The scatter plot shows the height and test scores of students in a math class. Describe the correlation between heights and test scores.

## 3. STOCKS
The prices of a technology stock over 5 days are shown in the table. Draw a scatter plot of the data and a line of fit.

<table>
<thead>
<tr>
<th>Day</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>8.30</td>
<td>8.60</td>
<td>8.55</td>
<td>8.90</td>
<td>9.30</td>
</tr>
</tbody>
</table>

## 4. ALGAE
One type of algae grows fastest at 31°C. The scatter plot shows data recording the amount of algae and the temperature of the water in various aquarium tanks. Draw a line of fit for this data and write a prediction equation. Will this prediction equation be accurate for temperatures above 31°C?

## 5. SPORTS
The scatter plot shows the height and score of different contestants shooting darts.

### a. What is the equation of the line of fit?

### b. What do you predict someone 5 feet tall would score?
Special Functions

Piecewise-Defined Functions A piecewise-defined function is written using two or more expressions. Its graph is often disjointed.

**Example**

Graph \( f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x - 1 & \text{if } x \geq 2 \end{cases} \).

First, graph the linear function \( f(x) = 2x \) for \( x < 2 \). Since 2 does not satisfy this inequality, stop with a circle at \((2, 4)\). Next, graph the linear function \( f(x) = x - 1 \) for \( x \geq 2 \). Since 2 does satisfy this inequality, begin with a dot at \((2, 1)\).

**Exercises**

Graph each function. Identify the domain and range.

1. \( f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 2x + 5 & \text{if } 0 \leq x \leq 2 \\ -x + 1 & \text{if } x > 2 \end{cases} \)

2. \( f(x) = \begin{cases} -x - 4 & \text{if } x < -7 \\ 5x - 1 & \text{if } -7 \leq x \leq 0 \\ 2x + 1 & \text{if } x > 0 \end{cases} \)

3. \( h(x) = \begin{cases} \frac{x}{3} & \text{if } x \leq 0 \\ 2x - 6 & \text{if } 0 < x < 2 \\ 1 & \text{if } x \geq 2 \end{cases} \)
### Special Functions

**Step Functions and Absolute Value Functions**

<table>
<thead>
<tr>
<th>Name</th>
<th>Written as</th>
<th>Graphed as</th>
</tr>
</thead>
<tbody>
<tr>
<td>Greatest Integer Function</td>
<td>( f(x) = [x] )</td>
<td><img src="image.png" alt="Graph" /></td>
</tr>
<tr>
<td>Absolute Value Function</td>
<td>( f(x) =</td>
<td>x</td>
</tr>
</tbody>
</table>

#### Example

**Graph** \( f(x) = 3|x| - 4 \).

Find several ordered pairs. Graph the points and connect them. You would expect the graph to look similar to its parent function, \( f(x) = |x| \).

| \( x \) | \( 3|x| - 4 \) |
|---------|----------------|
| 0       | -4             |
| 1       | -1             |
| 2       | 2              |
| -1      | -1             |
| -2      | 2              |

#### Exercises

Graph each function. Identify the domain and range.

1. \( f(x) = 2[x] \)
   ![Graph](image.png)

2. \( h(x) = |2x + 1| \)
   ![Graph](image.png)

3. \( f(x) = [x] + 4 \)
   ![Graph](image.png)
**2-6 Practice**

**Special Functions**

Graph each function. Identify the domain and range.

1. \( f(x) = \begin{cases} 
  x + 2 & \text{if } x \leq -2 \\
  3x & \text{if } x > -2 
\end{cases} \)

2. \( h(x) = \begin{cases} 
  4 - x & \text{if } x > 0 \\
  -2x - 2 & \text{if } x < 0 
\end{cases} \)

3. \( f(x) = \lfloor 0.5x \rfloor \)

4. \( f(x) = \lceil x \rceil - 2 \)

5. \( g(x) = -2|x| \)

6. \( f(x) = |x + 1| \)

7. **BUSINESS** *A Stitch in Time* charges $40 per hour or any fraction thereof for labor. Draw a graph of the step function that represents this situation.

![Labor Costs Graph](image)

8. **BUSINESS** A wholesaler charges a store $3.00 per pound for less than 20 pounds of candy and $2.50 per pound for 20 or more pounds. Draw a graph of the function that represents this situation.

![Candy Costs Graph](image)
1. **SAVINGS** Nathan puts $200 into a checking account as soon as he gets his paycheck. The value of his checking account is modeled by the formula \(200[m]\), where \(m\) is the number of months that Nathan has been working. After 105 days, how much money is in the account?

2. **FINANCE** A financial advisor handles the transactions for a customer. The median annual earnings for financial advisors is around $60,000. For every transaction, a certain financial advisor gets a 5% commission, regardless of whether the transaction is a deposit or withdrawal. Write a formula using the absolute value function for the advisor’s commission. Let \(D\) represent the value of one transaction.

3. **ROUNDING** A science teacher instructs students to round their measurements as follows: If a number is less than 0.5 of a millimeter, students are instructed to round down. If a number is exactly 0.5 or greater, students are told to round up to the next millimeter. Write a formula that takes a measurement \(x\) millimeters and yields the rounded off number.

4. **ARCHITECTURE** The cross-section of a roof is shown in the figure. Write an absolute value function that models the shape of the roof.

5. **GAMES** Some young people are playing a game where a wooden plank is used as a target. It is marked off into 6 equal parts. A value is written in each section to represent the score earned if the dart lands in that section. Let \(x\) denote the horizontal position of a dart on the board, where the center of the board is the origin. Negative values correspond to the left half of the dart board, and positive values correspond to the right half. A player’s score depends on the distance of the dart from the origin.

   a. Write a formula that gives the horizontal distance from the center of the dartboard.

   b. Write a formula using the greatest integer function that can be used to find the person’s score.
Parent Functions and Transformations

Parent Graphs The parent graph, which is the graph of the parent function, is the simplest of the graphs in a family. Each graph in a family of graphs has similar characteristics.

<table>
<thead>
<tr>
<th>Name</th>
<th>Characteristics</th>
<th>Parent Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant Function</td>
<td>Straight horizontal line</td>
<td>(y = a), where (a) is a real number</td>
</tr>
<tr>
<td>Linear Function</td>
<td>Straight diagonal line</td>
<td>Identity function (y = x)</td>
</tr>
<tr>
<td>Absolute Value Function</td>
<td>Diagonal lines shaped like a V</td>
<td>(y =</td>
</tr>
<tr>
<td>Quadratic Function</td>
<td>Curved like a parabola</td>
<td>(y = x^2)</td>
</tr>
</tbody>
</table>

Example Identify the type of function represented by each graph.

a. The graph is a diagonal line. The graph represents a linear function.

b. The graph is a parabolic curve. The graph represents a quadratic function.

Exercises

Identify the type of function represented by each graph.

1. 
2. 
3. 
4. 
5. 
6.
Transformations

Transformations of a parent graph may appear in a different location, may flip over an axis, or may appear to have been stretched or compressed.

Example

Describe the reflection in \( y = -|x| \). Then graph the function.

The graph of \( y = -|x| \) is a reflection of the graph of \( y = |x| \) in the \( x \)-axis.

Exercises

Describe the translation in each function. Then graph the function.

1. \( y = x - 4 \)
2. \( y = |x + 5| \)
3. \( y = x^2 - 3 \)

4. \( y = 5x \)
5. \( y = \frac{1}{2}|x| \)
6. \( y = 2x^2 \)
Describe the translation in each function. Then graph the function.

1. \( y = x + 3 \)

![Graph of \( y = x + 3 \)]

2. \( y = x^2 - 3 \)

![Graph of \( y = x^2 - 3 \)]

Describe the reflection in each function. Then graph the function.

3. \( y = (-x)^2 \)

![Graph of \( y = (-x)^2 \)]

4. \( y = -3 \)

![Graph of \( y = -3 \)]

Describe the dilation in each function. Then graph the function.

5. \( y = |2x| \)

![Graph of \( y = |2x| \)]

6. \( 4y = x^2 \)

![Graph of \( 4y = x^2 \)]

7. CHEMISTRY A scientist tested how fast a chemical reaction occurred at different temperatures. The data made this graph. What type of function shows the relation of temperature and speed of the chemical reaction?

![Graph of chemical reaction data]
2. ASTRONOMY The graph shows the velocity of the space probe Cassini as it passed Saturn. What type of function best models Cassini’s velocity?

3. GEOMETRY Chen made this graph to show how the perimeter of a square changes as the length of one side is increased. The original graph showed an identity function. How has it been dilated?

4. BUSINESS Maria earns an hourly wage of $10. She drew the following graph to show the relation of her income as a function of the hours she works. How did she modify the identity function to create her graph?

5. HOBBIES Laura launched a model rocket into the air. The height of her rocket over time is shown by the graph.

   a. What type of function does the graph show?

   b. In which axis has the function been reflected?

   c. Which directions has the graph been translated? How many units?

   d. What is the equation for the curve shown on the graph?
**3-2 Study Guide**  

**Solving Systems of Equations Algebraically**

**Substitution** To solve a system of linear equations by substitution, first solve for one variable in terms of the other in one of the equations. Then substitute this expression into the other equation and simplify.

**Example**  

Use substitution to solve the system of equations. \[2x - y = 9\] \[x + 3y = -6\]

Solve the first equation for \(y\) in terms of \(x\).

\[2x - y = 9\] First equation

\[-y = -2x + 9\] Subtract 2\(x\) from both sides.

\[y = 2x - 9\] Multiply both sides by -1.

Substitute the expression \(2x - 9\) for \(y\) into the second equation and solve for \(x\).

\[x + 3y = -6\] Second equation

\[x + 3(2x - 9) = -6\] Substitute \(2x - 9\) for \(y\).

\[x + 6x - 27 = -6\] Distributive Property

\[7x - 27 = -6\] Simplify.

\[7x = 21\] Add 27 to each side.

\[x = 3\] Divide each side by 7.

Now, substitute the value 3 for \(x\) in either original equation and solve for \(y\).

\[2x - y = 9\] First equation

\[2(3) - y = 9\] Replace \(x\) with 3.

\[6 - y = 9\] Simplify.

\[-y = 3\] Subtract 6 from each side.

\[y = -3\] Multiply each side by -1.

The solution of the system is (3, -3).

**Exercises**

Solve each system of equations by using substitution.

1. \[3x + y = 7\] \[4x + 2y = 16\]  
2. \[2x + y = 5\] \[3x - 3y = 3\]  
3. \[2x + 3y = -3\] \[x + 2y = 2\]

4. \[2x - y = 7\] \[6x - 3y = 14\]  
5. \[4x - 3y = 4\] \[2x + y = -8\]  
6. \[5x + y = 6\] \[3 - x = 0\]

7. \[x + 8y = -2\] \[x - 3y = 20\]  
8. \[2x - y = -4\] \[4x + y = 1\]  
9. \[x - y = -2\] \[2x - 3y = 2\]

10. \[x - 4y = 4\] \[2x + 12y = 13\]  
11. \[x + 3y = 2\] \[4x + 12y = 8\]  
12. \[2x + 2y = 4\] \[x - 2y = 0\]
**Solving Systems of Equations Algebraically**

**Elimination** To solve a system of linear equations by elimination, add or subtract the equations to eliminate one of the variables. You may first need to multiply one or both of the equations by a constant so that one of the variables has the opposite coefficient in one equation as it has in the other.

**Example 1**

**Use the elimination method to solve the system of equations.**

\[
\begin{align*}
2x - 4y &= -26 \\
3x - y &= -24
\end{align*}
\]

Multiply the second equation by \(-4\). Then add the equations to eliminate the \(y\) variable.

\[
\begin{align*}
2x - 4y &= -26 \\
3x - y &= -24 \quad \text{Multiply by} -4. \\
-10x + 4y &= 96 \\
x &= -7
\end{align*}
\]

Replace \(x\) with \(-7\) and solve for \(y\).

\[
\begin{align*}
2x - 4y &= -26 \\
2(-7) - 4y &= -26 \\
-14 - 4y &= -26 \\
-4y &= -12 \\
y &= 3
\end{align*}
\]

The solution is \((-7, 3)\).

**Example 2**

**Use the elimination method to solve the system of equations.**

\[
\begin{align*}
3x - 2y &= 4 \\
5x + 3y &= -25
\end{align*}
\]

Multiply the first equation by \(3\) and the second equation by \(2\). Then add the equations to eliminate the \(y\) variable.

\[
\begin{align*}
3x - 2y &= 4 \quad \text{Multiply by} 3. \\
9x - 6y &= 12 \\
10x + 6y &= -50 \quad \text{Multiply by} 2. \\
19x &= -38 \\
x &= -2
\end{align*}
\]

Replace \(x\) with \(-2\) and solve for \(y\).

\[
\begin{align*}
3x - 2y &= 4 \\
3(-2) - 2y &= 4 \\
-6 - 2y &= 4 \\
-2y &= 10 \\
y &= 5
\end{align*}
\]

The solution is \((-2, -5)\).

**Exercises**

Solve each system of equations by using elimination.

1. \[
\begin{align*}
2x - y &= 7 \\
3x + y &= 8
\end{align*}
\]

2. \[
\begin{align*}
x - 2y &= 4 \\
x + 6y &= 12
\end{align*}
\]

3. \[
\begin{align*}
3x + 4y &= -10 \\
x - 4y &= 2
\end{align*}
\]

4. \[
\begin{align*}
3x - y &= 12 \\
5x + 2y &= 20
\end{align*}
\]

5. \[
\begin{align*}
4x - y &= 6 \\
2x - \frac{y}{2} &= 4
\end{align*}
\]

6. \[
\begin{align*}
5x + 2y &= 12 \\
-6x - 2y &= -14
\end{align*}
\]

7. \[
\begin{align*}
2x + y &= 8 \\
3x + \frac{3}{2}y &= 12
\end{align*}
\]

8. \[
\begin{align*}
7x + 2y &= -1 \\
4y - 3y &= -13
\end{align*}
\]

9. \[
\begin{align*}
3x + 8y &= -6 \\
x - y &= 9
\end{align*}
\]

10. \[
\begin{align*}
5x + 4y &= 12 \\
7x - 6y &= 40
\end{align*}
\]

11. \[
\begin{align*}
-4x + y &= -12 \\
4x + 2y &= 6
\end{align*}
\]

12. \[
\begin{align*}
5x + 2y &= -8 \\
4x + 3y &= 2
\end{align*}
\]
3-2 Practice

Solving Systems of Equations Algebraically

Solve each system of equations by using substitution.

1. \(2x + y = 4\) \(3x + 2y = 1\)

2. \(x - 3y = 9\) \(x + 2y = -1\)

3. \(g + 3h = 8\) \(\frac{1}{3}g + h = 9\)

4. \(2a - 4b = 6\) \(-a + 2b = -3\)

5. \(2m + n = 6\) \(5m + 6n = 1\)

6. \(4x - 3y = -6\) \(-x - 2y = 7\)

7. \(u - 2v = \frac{1}{2}\) \(-u + 2v = 5\)

8. \(x - 3y = 16\) \(4x - y = 9\)

9. \(w + 3z = 1\) \(3w - 5z = -4\)

Solve each system of equations by using elimination.

10. \(2r + t = 5\) \(3r - t = 20\)

11. \(2m - n = -1\) \(3m + 2n = 30\)

12. \(6x + 3y = 6\) \(8x + 5y = 12\)

13. \(3j - k = 10\) \(4j - k = 16\)

14. \(2x - y = -4\) \(-4x + 2y = 6\)

15. \(2g + h = 6\) \(3g - 2h = 16\)

16. \(2t + 4v = 6\) \(-t - 2v = -3\)

17. \(3x - 2y = 12\) \(2x + \frac{2}{3}y = 14\)

18. \(\frac{1}{2}x + 3y = 11\) \(8x - 5y = 17\)

Solve each system of equations.

19. \(8x + 3y = -5\) \(10x + 6y = -13\)

20. \(8q - 15r = -40\) \(4q + 2r = 56\)

21. \(3x - 4y = 12\) \(\frac{1}{3}x - \frac{4}{9}y = \frac{4}{3}\)

22. \(4b - 2d = 5\) \(-2b + d = 1\)

23. \(x + 3y = 4\) \(x = 1\)

24. \(4m - 2p = 0\) \(-3m + 9p = 5\)

25. \(5g + 4k = 10\) \(-3g - 5k = 7\)

26. \(0.5x + 2y = 5\) \(x - 2y = -8\)

27. \(h - z = 3\) \(-3h + 3z = 6\)

28. **SPORTS** Last year the volleyball team paid $5 per pair for socks and $17 per pair for shorts on a total purchase of $315. This year they spent $342 to buy the same number of pairs of socks and shorts because the socks now cost $6 a pair and the shorts cost $18.

a. Write a system of two equations that represents the number of pairs of socks and shorts bought each year.

b. How many pairs of socks and shorts did the team buy each year?
1. SUPPLIES Kirsta and Arthur both need pens and blank CDs. The equation that represents Kirsta’s purchases is \( y = 27 - 3x \). The equation that represents Arthur’s purchases is \( y = 17 - x \). If \( x \) represents the price of the pens, and \( y \) represents the price of the CDs, what are the prices of the pens and the CDs?

2. WALKING Amy is walking a straight path that can be represented by the equation \( y = 2x + 3 \). At the same time Kendra is walking the straight path that has the equation \( 3y = 6x + 6 \). What is the solution to the system of equations that represents the paths the two girls walked? Explain.

3. CAFETERIA To furnish a cafeteria, a school can spend $5200 on tables and chairs. Tables cost $200 and chairs cost $40. Each table will have 8 chairs around it. How many tables and chairs will the school purchase?

4. PRICES At a store, toothbrushes cost \( x \) dollars and bars of soap cost \( y \) dollars. One customer bought 2 toothbrushes and 1 bar of soap for $11. Another customer bought 6 toothbrushes and 5 bars of soap for $38. Both amounts do not include tax. Write and solve a system of equations for \( x \) and \( y \).

5. GAMES Mark and Stephanie are playing darts according to World Darts Federation rules. Each toss earns points depending on where the dart lands. Each ring and sector is worth a different number of points. Mark and Stephanie both toss three darts in the first round. It happens that their darts only land in two areas of the board.

   a. Stephanie earned a total of 60 points with two darts landing in the first area and one dart landing in the second area. One of Mark’s darts landed in the first area and two in the second. He scored 75 points. Write a system of equations for their scores. Use \( x \) for the first area and \( y \) for the second area.

   b. Solve the equations. How many points is the first area worth? How many points is the second area worth?

   c. If Mark and Stephanie are playing to 150 points and their darts continue to land in only the two areas, which different combinations of darts landing in the two areas will add up to 150? How many darts landing in the first area would total 150 points? How many darts landing in the second area?
Optimization with Linear Programming

Maximum and Minimum Values  When a system of linear inequalities produces a bounded polygonal region, the maximum or minimum value of a related function will occur at a vertex of the region.

**Example**  Graph the system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the function $f(x, y) = 3x + 2y$ for this polygonal region.

\[
\begin{align*}
y &\leq 4 \\
y &\leq -x + 6 \\
y &\geq \frac{1}{2}x - \frac{3}{2} \\
y &\leq 6x + 4
\end{align*}
\]

First find the vertices of the bounded region. Graph the inequalities.

The polygon formed is a quadrilateral with vertices at $(0, 4), (2, 4), (5, 1), \text{ and } (-1, -2)$. Use the table to find the maximum and minimum values of $f(x, y) = 3x + 2y$.

<table>
<thead>
<tr>
<th>$(x, y)$</th>
<th>$3x + 2y$</th>
<th>$f(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 4)$</td>
<td>$3(0) + 2(4)$</td>
<td>$8$</td>
</tr>
<tr>
<td>$(2, 4)$</td>
<td>$3(2) + 2(4)$</td>
<td>$14$</td>
</tr>
<tr>
<td>$(5, 1)$</td>
<td>$3(5) + 2(1)$</td>
<td>$17$</td>
</tr>
<tr>
<td>$(-1, -2)$</td>
<td>$3(-1) + 2(-2)$</td>
<td>$-7$</td>
</tr>
</tbody>
</table>

The maximum value is 17 at $(5, 1)$. The minimum value is $-7$ at $(-1, -2)$.

**Exercises**

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

1. $y \geq 2$
   
   $1 \leq x \leq 5$
   
   $y \leq x + 3$
   
   $f(x, y) = 3x - 2y$

2. $y \geq -2$
   
   $y \geq 2x - 4$
   
   $x - 2y \geq -1$
   
   $f(x, y) = 4x - y$

3. $x + y \geq 2$
   
   $4y \leq x + 8$
   
   $y \geq 2x - 5$
   
   $f(x, y) = 4x + 3y$
**Optimization with Linear Programming**

**Optimization**  When solving linear programming problems, use the following procedure.

1. Define variables.
2. Write a system of inequalities.
3. Graph the system of inequalities.
4. Find the coordinates of the vertices of the feasible region.
5. Write an expression to be maximized or minimized.
6. Substitute the coordinates of the vertices in the expression.
7. Select the greatest or least result to answer the problem.

**Example**  A painter has exactly 32 units of yellow dye and 54 units of green dye. He plans to mix as many gallons as possible of color A and color B. Each gallon of color A requires 4 units of yellow dye and 1 unit of green dye. Each gallon of color B requires 1 unit of yellow dye and 6 units of green dye. Find the maximum number of gallons he can mix.

**Step 1**  Define the variables.

- \( x \) = the number of gallons of color A made
- \( y \) = the number of gallons of color B made

**Step 2**  Write a system of inequalities.

Since the number of gallons made cannot be negative, \( x \geq 0 \) and \( y \geq 0 \).

There are 32 units of yellow dye; each gallon of color A requires 4 units, and each gallon of color B requires 1 unit.

So \( 4x + y \leq 32 \).

Similarly for the green dye, \( x + 6y \leq 54 \).

**Steps 3 and 4**  Graph the system of inequalities and find the coordinates of the vertices of the feasible region. The vertices of the feasible region are \((0, 0)\), \((0, 9)\), \((6, 8)\), and \((8, 0)\).

**Steps 5–7**  Find the maximum number of gallons, \( x + y \), that he can make. The maximum number of gallons the painter can make is 14, 6 gallons of color A and 8 gallons of color B.

**Exercises**

1. **FOOD**  A delicatessen has 12 pounds of plain sausage and 10 pounds of spicy sausage.
   
   A pound of Bratwurst A contains \( \frac{3}{4} \) pound of plain sausage and \( \frac{1}{4} \) pound of spicy sausage. A pound of Bratwurst B contains \( \frac{1}{2} \) pound of each sausage.
   
   Find the maximum number of pounds of bratwurst that can be made.

2. **MANUFACTURING**  Machine A can produce 30 steering wheels per hour at a cost of $8 per hour. Machine B can produce 40 steering wheels per hour at a cost of $12 per hour. The company can use either machine by itself or both machines at the same time. What is the minimum number of hours needed to produce 380 steering wheels if the cost must be no more than $108?
Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

1. \(2x - 4 \leq y\)
   \(-2x - 4 \leq y\)
   \(y \leq 2\)
   \(f(x, y) = -2x + y\)

2. \(3x - y \leq 7\)
   \(2x - y \geq 3\)
   \(y \geq x - 3\)
   \(f(x, y) = x - 4y\)

3. \(x \geq 0\)
   \(y \geq 0\)
   \(y \leq 6\)
   \(f(x, y) = 3x + y\)

4. \(x \leq 0\)
   \(y \geq 0\)
   \(4x + y \geq -7\)
   \(f(x, y) = -x - 4y\)

5. \(y \leq 3x + 6\)
   \(4y + 3x \leq 3\)
   \(x \geq -2\)
   \(f(x, y) = -x + 3y\)

6. \(2x + 3y \geq 6\)
   \(2x - y \leq 2\)
   \(x \geq 0\)
   \(y \geq 0\)
   \(f(x, y) = x + 4y + 3\)

7. **Production** A glass blower can form 8 simple vases or 2 elaborate vases in an hour. In a work shift of no more than 8 hours, the worker must form at least 40 vases.
   
   a. Let \(s\) represent the hours forming simple vases and \(e\) the hours forming elaborate vases. Write a system of inequalities involving the time spent on each type of vase.
   
   b. If the glass blower makes a profit of $30 per hour worked on the simple vases and $35 per hour worked on the elaborate vases, write a function for the total profit on the vases.
   
   c. Find the number of hours the worker should spend on each type of vase to maximize profit. What is that profit?
1. **REGIONS** A region in the plane is formed by the equations \( x - y < 3 \), \( x - y > -3 \), and \( x + y > -3 \). Is this region bounded or unbounded? Explain.

2. **MANUFACTURING** Eighty workers are available to assemble tables and chairs. It takes 5 people to assemble a table and 3 people to assemble a chair. The workers always make at least as many tables as chairs because the tables are easier to make. If \( x \) is the number of tables and \( y \) is the number of chairs, the system of inequalities that represent what can be assembled is \( x > 0 \), \( y > 0 \), \( y \leq x \), and \( 5x + 3y \leq 80 \). What is the maximum total number of chairs and tables the workers can make?

3. **FISH** An aquarium is 7000 cubic inches. Nathan wants to populate the aquarium with neon tetras and catfish. It is recommended that each neon tetra be allowed 170 cubic inches and each catfish be allowed 700 cubic inches of space. Nathan would like at least one catfish for every 4 neon tetras. Let \( n \) be the number of neon tetra and \( c \) be the number of catfish. The following inequalities form the feasible region for this situation: \( n > 0 \), \( c > 0 \), \( 4c \geq n \), and \( 170n + 700c \leq 7000 \). What is the maximum number of fish Nathan can put in his aquarium?

4. **ELEVATION** A trapezoidal park is built on a slight incline. The function for the ground elevation above sea level is \( f(x, y) = x - 3y + 20 \) feet. What are the coordinates of the highest point in the park?

5. **CERAMICS** Josh has 8 days to make pots and plates to sell at a local fair. Each pot weighs 2 pounds and each plate weighs 1 pound. Josh cannot carry more than 50 pounds to the fair. Each day, he can make at most 5 plates and at most 3 pots. He will make $12 profit for every plate and $25 profit for every pot that he sells.

   a. Write linear inequalities to represent the number of pots \( p \) and plates \( a \) Josh may bring to the fair.

   b. List the coordinates of the vertices of the feasible region.

   c. How many pots and how many plates should Josh make to maximize his potential profit?
A matrix can be described by its **dimensions**. A matrix with \( m \) rows and \( n \) columns is an \( m \times n \) matrix.

### Example 1

Owls’ eggs incubate for 30 days and their fledgling period is also 30 days. Swifts’ eggs incubate for 20 days and their fledgling period is 44 days. Pigeon eggs incubate for 15 days, and their fledgling period is 17 days. Eggs of the king penguin incubate for 53 days, and the fledgling time for a king penguin is 360 days. Write a \( 2 \times 4 \) matrix to organize this information. **Source:** *The Cambridge Factfinder*

<table>
<thead>
<tr>
<th>Incubation</th>
<th>Owl</th>
<th>Swift</th>
<th>Pigeon</th>
<th>King Penguin</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>30</td>
<td>20</td>
<td>15</td>
<td>53</td>
</tr>
<tr>
<td>Fledgling</td>
<td>30</td>
<td>44</td>
<td>17</td>
<td>360</td>
</tr>
</tbody>
</table>

### Example 2

What are the dimensions of matrix \( A \) if \( A = \begin{bmatrix} 13 & 10 & -3 & 45 \\ 2 & 8 & 15 & 80 \end{bmatrix} \)?

Since matrix \( A \) has 2 rows and 4 columns, the dimensions of \( A \) are \( 2 \times 4 \).

### Exercises

**State the dimensions of each matrix.**

1. \[
\begin{bmatrix}
15 & 5 & 27 & -4 \\
23 & 6 & 0 & 5 \\
14 & 70 & 24 & -3 \\
63 & 3 & 42 & 90
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
16 & 12 & 0
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
71 & 44 \\
39 & 27 \\
45 & 16 \\
92 & 53 \\
78 & 65
\end{bmatrix}
\]

4. A travel agent provides for potential travelers the normal high temperatures for the months of January, April, July, and October for various cities. In Boston these figures are 36°, 56°, 82°, and 63°. In Dallas they are 54°, 76°, 97°, and 79°. In Los Angeles they are 68°, 72°, 84°, and 79°. In Seattle they are 46°, 58°, 74°, and 60°. In St. Louis they are 38°, 67°, 89°, and 69°. Organize this information in a \( 4 \times 5 \) matrix. **Source:** *The New York Times Almanac*
Introduction to Matrices

Elements of a Matrix  A matrix is a rectangular array of variables or constants in horizontal rows and vertical columns. The values are called elements and are identified by their location in the matrix. The location of an element is written as a subscript with the number of its row followed by the number of its column. For example, \( a_{12} \) is the element in the first row and second column of matrix \( A \).

In the matrices below, 11 is the value of \( a_{12} \) in the first matrix. The value of \( b_{32} \) in the second matrix is 7.

\[
A = \begin{bmatrix}
7 & 11 & 2 & 8 \\
5 & 4 & 10 & 1 \\
9 & 3 & 6 & 12
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
3 & 9 & 12 \\
5 & 10 & 15 \\
8 & 7 & 6 \\
11 & 13 & 1 \\
4 & 2 & 14
\end{bmatrix}
\]

Example 1  Find the value of \( c_{23} \).

\[
C = \begin{bmatrix}
2 & 5 & 3 \\
3 & 4 & 1
\end{bmatrix}
\]

Since \( c_{23} \) is the element in row 2, column 3, the value of \( c_{23} \) is 1.

Example 2  Find the value of \( d_{54} \).

\[
\text{matrix } D = \begin{bmatrix}
25 & 11 & 4 & 1 & 20 \\
7 & 8 & 9 & 12 & 13 \\
17 & 6 & 15 & 18 & 2 \\
22 & 16 & 21 & 24 & 19 \\
5 & 23 & 3 & 14 & 10
\end{bmatrix}
\]

Since \( d_{54} \) is the element in row 5, column 4, the value of \( d_{54} \) is 14.

Exercises

Identify each element for the following matrices.

\[
F = \begin{bmatrix}
12 & 7 & 5 \\
9 & 2 & 11 \\
6 & 14 & 8 \\
1 & 4 & 3
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
1 & 14 & 13 & 12 \\
2 & 15 & 20 & 11 \\
3 & 16 & 19 & 10 \\
4 & 17 & 18 & 9 \\
5 & 6 & 7 & 8
\end{bmatrix}
\]

\[
H = \begin{bmatrix}
5 & 9 & 11 & 4 \\
3 & 7 & 2 & 10 \\
8 & 2 & 6 & 1
\end{bmatrix}
\]

1. \( f_{32} \)  
2. \( g_{51} \)  
3. \( h_{22} \)  
4. \( g_{43} \)  
5. \( h_{34} \)  
6. \( f_{23} \)  
7. \( h_{14} \)  
8. \( f_{42} \)  
9. \( g_{14} \)
4-1 Practice

Introduction to Matrices

State the dimensions of each matrix.

1. $[-3 \ -3 \ 7]$
2. $\begin{bmatrix} 5 & 8 & -1 \\ -2 & 1 & 8 \end{bmatrix}$
3. $\begin{bmatrix} -2 & 2 & -2 & 3 \\ 5 & 16 & 0 & 0 \\ 4 & 7 & -1 & 4 \end{bmatrix}$

Identify each element for the following matrices.

$$A = \begin{bmatrix} 4 & 7 & 0 \\ 9 & 8 & -4 \\ 3 & 0 & 5 \\ -1 & 2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 6 & -1 & 0 \\ 9 & 5 & 7 & 2 \end{bmatrix}.$$  

4. $b_{23}$  
5. $a_{42}$  
6. $b_{11}$

7. $a_{32}$  
8. $b_{14}$  
9. $a_{23}$

10. **TICKET PRICES** The table at the right gives ticket prices for a concert. Write a $2 \times 3$ matrix that represents the cost of a ticket.

<table>
<thead>
<tr>
<th></th>
<th>Child</th>
<th>Student</th>
<th>Adult</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cost Purchased</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>in Advance</strong></td>
<td>$6$</td>
<td>$12$</td>
<td>$18$</td>
</tr>
<tr>
<td><strong>Cost Purchased at</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>the Door</strong></td>
<td>$8$</td>
<td>$15$</td>
<td>$22$</td>
</tr>
</tbody>
</table>

11. **CONSTRUCTION** During each of the last three weeks, a road-building crew has used three truckloads of gravel. The table at the right shows the amount of gravel in each load.

a. Write a matrix for the amount of gravel in each load.

b. What are the dimensions of the matrix?
1. HAWAII  The table shows the population and area of some of the islands in Hawaii. What would be the dimensions of a matrix that represented this information?

<table>
<thead>
<tr>
<th>Island</th>
<th>Population</th>
<th>Area(mi²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hawaii</td>
<td>120,317</td>
<td>4038</td>
</tr>
<tr>
<td>Maui</td>
<td>91,361</td>
<td>729</td>
</tr>
<tr>
<td>Oahu</td>
<td>836,231</td>
<td>594</td>
</tr>
<tr>
<td>Kauai</td>
<td>50,947</td>
<td>549</td>
</tr>
<tr>
<td>Lanai</td>
<td>2426</td>
<td>140</td>
</tr>
</tbody>
</table>

Source: Virtual Tour of Hawaii

2. LAUNDRY  Carl is looking for a Laundromat. SuperWash has 20 small washers, 10 large washers, and 20 dryers. QuickClean has 40 small washers, 5 large washers, and 50 dryers. ToughSuds has 15 small washers, 40 large washers, and 100 dryers. Write a matrix to organize this information.

3. CITY DISTANCES  The incomplete matrix shown gives the approximate distances between Chicago, Los Angeles, and New York City. Complete the matrix.

<table>
<thead>
<tr>
<th></th>
<th>NYC</th>
<th>Chicago</th>
<th>Los Angeles</th>
</tr>
</thead>
<tbody>
<tr>
<td>NYC</td>
<td>0</td>
<td>2790</td>
<td></td>
</tr>
<tr>
<td>Chicago</td>
<td>810</td>
<td>2050</td>
<td></td>
</tr>
</tbody>
</table>

4. INVENTORY  A store manager records the number of light bulbs in stock for 3 different brands over a five-day period. The manager decides to make a matrix of this information. Each row represents a different brand, and each column represents a different day. The entry in column $N$ represents the inventories at the beginning of day $N$.

\[
\begin{bmatrix}
25 & 24 & 22 & 20 & 19 \\
30 & 27 & 25 & 22 & 21 \\
28 & 25 & 21 & 19 & 19 \\
\end{bmatrix}
\]

Assuming that the inventories were never replenished, which brand holds the record for most light bulbs sold on a given day?

5. SHOE SALES  A shoe store manager keeps track of the amount of money made by each of three salespeople for each day of a workweek. Monday through Friday, Carla made $40, $70, $35, $50, and $20. John made $30, $60, $20, $45, and $30. Mary made $35, $90, $30, $40, and $30.

a. Organize this data in a 3 by 5 matrix.

b. Which salesperson made the most money that week?
Add and Subtract Matrices Matrices with the same dimensions can be added together or one can be subtracted from the other.

### Addition of Matrices

\[
\begin{bmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{bmatrix} + \begin{bmatrix}
  j & k & l \\
  m & n & o \\
  p & q & r
\end{bmatrix} = \begin{bmatrix}
  a+j & b+k & c+l \\
  d+m & e+n & f+o \\
  g+p & h+q & i+r
\end{bmatrix}
\]

### Subtraction of Matrices

\[
\begin{bmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{bmatrix} - \begin{bmatrix}
  j & k & l \\
  m & n & o \\
  p & q & r
\end{bmatrix} = \begin{bmatrix}
  a-j & b-k & c-l \\
  d-m & e-n & f-o \\
  g-p & h-q & i-r
\end{bmatrix}
\]

#### Example 1

Find \( A + B \) if \( A = \begin{bmatrix} 6 & -7 \\ 2 & -12 \end{bmatrix} \) and \( B = \begin{bmatrix} 4 & 2 \\ -5 & -6 \end{bmatrix} \).

\[
A + B = \begin{bmatrix} 6 & -7 \\ 2 & -12 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -5 & -6 \end{bmatrix} = \begin{bmatrix} 10 & -5 \\ -3 & -18 \end{bmatrix}
\]

#### Example 2

Find \( A - B \) if \( A = \begin{bmatrix} -2 & 8 \\ 3 & -4 \\ 10 & 7 \end{bmatrix} \) and \( B = \begin{bmatrix} 4 & -3 \\ -2 & 1 \\ -6 & 8 \end{bmatrix} \).

\[
A - B = \begin{bmatrix} -2 & 8 \\ 3 & -4 \\ 10 & 7 \end{bmatrix} - \begin{bmatrix} 4 & -3 \\ -2 & 1 \\ -6 & 8 \end{bmatrix} = \begin{bmatrix} -6 & 11 \\ 15 & -3 \\ 16 & -1 \end{bmatrix}
\]

### Exercises

Perform the indicated operations. If the matrix does not exist, write \textit{impossible}.

1. \[ \begin{bmatrix} 8 & 7 \\ -10 & -6 \end{bmatrix} - \begin{bmatrix} -4 & 3 \\ 2 & -12 \end{bmatrix} \]
2. \[ \begin{bmatrix} 6 & -5 & 9 \\ -3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} -4 & 3 & 2 \\ 6 & 9 & -4 \end{bmatrix} \]
3. \[ \begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -6 & 3 & -2 \end{bmatrix} \]
4. \[ \begin{bmatrix} 5 & -2 \\ -4 & 6 \\ 7 & 9 \end{bmatrix} + \begin{bmatrix} -11 & 6 \\ 2 & -5 \\ -4 & -7 \end{bmatrix} \]
5. \[ \begin{bmatrix} 8 & 0 & -6 \\ 4 & 5 & -11 \\ -7 & 3 & 4 \end{bmatrix} - \begin{bmatrix} -2 & 1 & 7 \\ 3 & -4 & 3 \\ -8 & 5 & 6 \end{bmatrix} \]
6. \[ \begin{bmatrix} 3 & 2 \\ 4 & 5 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \]
**Operations with Matrices**

**Scalar Multiplication** You can multiply an $m \times n$ matrix by a scalar $k$.

Scalar Multiplication \[ k \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \end{bmatrix} \]

**Example** If \( A = \begin{bmatrix} 4 & 0 \\ -6 & 3 \end{bmatrix} \) and \( B = \begin{bmatrix} -1 & 5 \\ 7 & 8 \end{bmatrix} \), find \( 3B - 2A \).

\[
3B - 2A = 3\begin{bmatrix} -1 & 5 \\ 7 & 8 \end{bmatrix} - 2\begin{bmatrix} 4 & 0 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} 3(-1) & 3(5) \\ 3(7) & 3(8) \end{bmatrix} - \begin{bmatrix} 2(4) & 2(0) \\ 2(-6) & 2(3) \end{bmatrix} = \begin{bmatrix} -3 & 15 \\ 21 & 24 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ -12 & 6 \end{bmatrix} = \begin{bmatrix} -3 - 8 & 15 - 0 \\ 21 - (-12) & 24 - 6 \end{bmatrix} = \begin{bmatrix} -11 & 15 \\ 33 & 18 \end{bmatrix}
\]

**Exercises**

Perform the indicated operations. If the matrix does not exist, write *impossible*.

1. \( \begin{bmatrix} 2 & -5 & 3 \\ 0 & 7 & -1 \\ -4 & 6 & 9 \end{bmatrix} \)

2. \( -\frac{1}{3} \begin{bmatrix} 6 & 15 & 9 \\ 51 & -33 & 24 \\ -18 & 3 & 45 \end{bmatrix} \)

3. \( 0.2 \begin{bmatrix} 25 & -10 & -45 \\ 5 & 55 & -30 \\ 60 & 35 & -95 \end{bmatrix} \)

4. \( 3 \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix} - 2 \begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix} \)

5. \( -2 \begin{bmatrix} 3 & -1 \\ 0 & 7 \end{bmatrix} + 4 \begin{bmatrix} -2 & 0 \\ 2 & 5 \end{bmatrix} \)

6. \( 2 \begin{bmatrix} 6 & -10 \\ -5 & 8 \end{bmatrix} + 5 \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \)

7. \( 4 \begin{bmatrix} 1 & -2 & 5 \\ -3 & 4 & 1 \end{bmatrix} - 2 \begin{bmatrix} 4 & 3 & -4 \\ 2 & -5 & -1 \end{bmatrix} \)

8. \( 8 \begin{bmatrix} 2 & 1 \\ 3 & -1 \\ -2 & 4 \end{bmatrix} + 3 \begin{bmatrix} 4 & 0 \\ -2 & 3 \\ 3 & -4 \end{bmatrix} \)

9. \( \frac{1}{4} \begin{bmatrix} 9 & 1 \\ -7 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -5 \\ 1 & 7 \end{bmatrix} \)
4-2 Practice

Operations with Matrices

Perform the indicated operations. If the matrix does not exist, write impossible.

1. \[
\begin{bmatrix}
2 & -1 \\
3 & 7 \\
14 & -9
\end{bmatrix}
+ \begin{bmatrix}
-6 & 9 \\
7 & -11 \\
-8 & 17
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
4 \\
-71 \\
18
\end{bmatrix}
- \begin{bmatrix}
-67 \\
45 \\
-24
\end{bmatrix}
\]

3. \[
-3\begin{bmatrix}
-1 & 0 \\
17 & -11
\end{bmatrix}
+ 4\begin{bmatrix}
-3 & 16 \\
-21 & 12
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
2 & -1 & 8 \\
4 & 7 & 9
\end{bmatrix}
- 2\begin{bmatrix}
1 & 4 & -3 \\
7 & 2 & -6
\end{bmatrix}
\]

5. \[
-2\begin{bmatrix}
1 \\
2
\end{bmatrix}
+ 4\begin{bmatrix}
0 \\
5
\end{bmatrix}
- \begin{bmatrix}
10 \\
18
\end{bmatrix}
\]

6. \[
\frac{3}{4}\begin{bmatrix}
8 & 12 \\
-16 & 20
\end{bmatrix}
+ \frac{2}{3}\begin{bmatrix}
27 & -9 \\
54 & -18
\end{bmatrix}
\]

Use matrices \(A = \begin{bmatrix}
4 & -1 & 0 \\
-3 & 6 & 2
\end{bmatrix}\), \(B = \begin{bmatrix}
-2 & 4 & 5 \\
1 & 0 & 9
\end{bmatrix}\), and \(C = \begin{bmatrix}
10 & -8 & 6 \\
-6 & -4 & 20
\end{bmatrix}\) to find the following.

7. \(A - B\)

8. \(A - C\)

9. \(-3B\)

10. \(4B - A\)

11. \(-2B - 3C\)

12. \(A + 0.5C\)

13. ECONOMICS Use the table that shows loans by an economic development board to women and men starting new businesses.

<table>
<thead>
<tr>
<th></th>
<th>Women</th>
<th>Men</th>
</tr>
</thead>
<tbody>
<tr>
<td>Businesses</td>
<td>Loan Amount ($)</td>
<td>Businesses</td>
</tr>
<tr>
<td>2008</td>
<td>27 $567,000</td>
<td>36 $864,000</td>
</tr>
<tr>
<td>2009</td>
<td>41 $902,000</td>
<td>32 $672,000</td>
</tr>
<tr>
<td>2010</td>
<td>35 $777,000</td>
<td>28 $562,000</td>
</tr>
</tbody>
</table>

a. Write two matrices that represent the number of new businesses and loan amounts, one for women and one for men.

b. Find the sum of the numbers of new businesses and loan amounts for both men and women over the three-year period expressed as a matrix.

14. PET NUTRITION Use the table that gives nutritional information for two types of dog food. Find the difference in the percent of protein, fat, and fiber between Mix B and Mix A expressed as a matrix.

<table>
<thead>
<tr>
<th></th>
<th>% Protein</th>
<th>% Fat</th>
<th>% Fiber</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mix A</td>
<td>22</td>
<td>12</td>
<td>5</td>
</tr>
<tr>
<td>Mix B</td>
<td>24</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

\[
\begin{bmatrix}
20 & 12 \\
12 & 8.5
\end{bmatrix}
\]

What can you do to this matrix in order to create another matrix that represents fares for 5 people?

**SOURCE:** Museum of Modern Art

2. **NEGATION** Two engineers need to negate all the entries of a matrix. One engineer tries to do this by multiplying the matrix by \(-1\). The other engineer tries to do this by subtracting twice the matrix from itself. Which engineer, if either, will get the correct result?

3. **PLANE FARES** The airfares for travel between New York, Chicago, and Los Angeles are organized in the first matrix. The second matrix gives the tax surcharges for corresponding flights.

\[
\begin{bmatrix}
NYC & Chicago & Los Angeles \\
NYC & 0 & 440 & 700 \\
Chicago & 460 & 0 & 660 \\
Los Angeles & 850 & 700 & 0
\end{bmatrix}
\]

Write a matrix that represents the full cost for travel between these cities.

4. **SUNFLOWERS** Matrix \(H\) is a 3 by 1 matrix that contains the initial heights of three sunflowers. Matrix \(G\) is a 3 by 1 matrix that contains the numbers of inches the corresponding sunflowers grow in a week. What does matrix \(H + 4G\) represent?

5. **DINNER** The menu shows prices for some dishes at a restaurant.

**Il Ristorante Menu**

<table>
<thead>
<tr>
<th></th>
<th>Regular</th>
<th>Half-portion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lamb</td>
<td>$17.00</td>
<td>$9.00</td>
</tr>
<tr>
<td>Chicken</td>
<td>$14.00</td>
<td>$7.00</td>
</tr>
<tr>
<td>Steak</td>
<td>$22.00</td>
<td>$11.00</td>
</tr>
</tbody>
</table>

a. Make a 3 by 2 matrix to organize these data.

b. Let \(M\) be the matrix you wrote for part a. Write an expression involving \(M\) that would give prices that include an additional 20% to cover tax and tip.

c. Compute the matrix you described in part b.
**Multiply Matrices**

You can multiply two matrices if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

<table>
<thead>
<tr>
<th>Multiplication of Matrices</th>
<th>$A \cdot B = AB$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{bmatrix} a &amp; b \ c &amp; d \end{bmatrix} \cdot \begin{bmatrix} e &amp; f \ g &amp; h \end{bmatrix}$</td>
<td>$\begin{bmatrix} ae + bg &amp; af + bh \ ce + dg &amp; cf + dh \end{bmatrix}$</td>
</tr>
</tbody>
</table>

**Example**

Find $AB$ if $A = \begin{bmatrix} -4 & 3 \\ 2 & -2 \\ 1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 \\ -1 & 3 \end{bmatrix}$.

$AB = \begin{bmatrix} -4 & 3 \\ 2 & -2 \\ 1 & 7 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2 \\ -1 & 3 \end{bmatrix}$

Substitution

$= \begin{bmatrix} -4(5) + 3(-1) & -4(-2) + 3(3) \\ 2(5) + (-2)(-1) & 2(-2) + (-2)(3) \\ 1(5) + 7(-1) & 1(-2) + 7(3) \end{bmatrix}$

Multiply columns by rows.

$= \begin{bmatrix} -23 & 17 \\ 12 & -10 \\ -2 & 19 \end{bmatrix}$

Simplify.

**Exercises**

Find each product, if possible.

1. $\begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

2. $\begin{bmatrix} -1 & 0 \\ 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$

3. $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 1 \\ 5 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 3 & -2 \\ 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 5 & -2 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}$

7. $\begin{bmatrix} 6 & 10 \\ -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 4 & -3 \\ -2 & 7 \end{bmatrix}$

8. $\begin{bmatrix} 7 & -2 \\ 5 & -4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 \\ -2 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 \\ 3 & 1 \\ -1 & 3 & 1 \end{bmatrix}$
Multiplying Matrices

Multiplicative Properties  The Commutative Property of Multiplication does not hold for matrices.

<table>
<thead>
<tr>
<th>Properties of Matrix Multiplication</th>
<th>For any matrices $A$, $B$, and $C$ for which the matrix product is defined, and any scalar $c$, the following properties are true.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Associative Property of Matrix Multiplication</td>
<td>$(AB)C = A(BC)$</td>
</tr>
<tr>
<td>Associative Property of Scalar Multiplication</td>
<td>$c(AB) = (cA)B = A(cB)$</td>
</tr>
<tr>
<td>Left Distributive Property</td>
<td>$C(A + B) = CA + CB$</td>
</tr>
<tr>
<td>Right Distributive Property</td>
<td>$(A + B)C = AC + BC$</td>
</tr>
</tbody>
</table>

**Example**

Use $A = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 5 & -3 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & -2 \\ 6 & 3 \end{bmatrix}$ to find each product.

a. $(A + B)C$

$(A + B)C = \left( \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 5 & -3 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 & -2 \\ 6 & 3 \end{bmatrix}$

$= \begin{bmatrix} 6 & -3 \\ 7 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 6 & 3 \end{bmatrix}$

$= \begin{bmatrix} 6(1) + (-3)(6) & 6(-2) + (-3)(3) \\ 7(1) + (-2)(6) & 7(-2) + (-2)(3) \end{bmatrix}$

$= \begin{bmatrix} -12 & -21 \\ -5 & -20 \end{bmatrix}$

b. $AC + BC$

$AC + BC = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 6 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 5 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 6 & 3 \end{bmatrix}$

$= \begin{bmatrix} 4(1) + (-3)(6) & 4(-2) + (-3)(3) \\ 2(1) + 1(6) & 2(-2) + 1(3) \end{bmatrix} + \begin{bmatrix} 2(1) + 0(6) & 2(-2) + 0(3) \\ 5(1) + (-3)(6) & 5(-2) + (-3)(3) \end{bmatrix}$

$= \begin{bmatrix} -14 & -17 \\ 8 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ -13 & -19 \end{bmatrix} = \begin{bmatrix} -12 & -21 \\ -5 & -20 \end{bmatrix}$

Note that although the results in the example illustrate the Right Distributive Property, they do not prove it.

**Exercises**

Use $A = \begin{bmatrix} 3 & 2 \\ 5 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 4 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -1/2 & -2 \\ 1 & -3 \end{bmatrix}$, and scalar $c = -4$ to determine whether the following equations are true for the given matrices.

1. $c(AB) = (cA)B$
2. $AB = BA$
3. $BC = CB$
4. $(AB)C = A(BC)$
5. $C(A + B) = AC + BC$
6. $c(A + B) = cA + cB$
4-3 Practice

Multiplying Matrices

Determine whether each matrix product is defined. If so, state the dimensions of the product.

1. \(A_{7 \times 4} \cdot B_{4 \times 3}\)  
2. \(A_{3 \times 5} \cdot M_{5 \times 8}\)  
3. \(M_{2 \times 1} \cdot A_{1 \times 6}\)  
4. \(M_{3 \times 2} \cdot A_{3 \times 2}\)  
5. \(P_{1 \times 9} \cdot Q_{9 \times 1}\)  
6. \(P_{9 \times 1} \cdot Q_{1 \times 9}\)

Find each product, if possible.

7. \[
\begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 & 7 \\ 6 & 0 & -5 \end{bmatrix}
\]
8. \[
\begin{bmatrix} 2 & 4 \\ 7 & -1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 0 \\ 2 & 5 \end{bmatrix}
\]
9. \[
\begin{bmatrix} -3 & 0 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 7 & -1 \end{bmatrix}
\]
10. \[
\begin{bmatrix} 3 & -2 & 7 \\ 6 & 0 & -5 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 & 7 \\ 6 & 0 & -5 \end{bmatrix}
\]
11. \[
\begin{bmatrix} 4 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}
\]
12. \[
\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 2 \end{bmatrix}
\]
13. \[
\begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}
\]
14. \[
\begin{bmatrix} -15 & -9 \end{bmatrix} \cdot \begin{bmatrix} 6 & 11 \\ 23 & -10 \end{bmatrix}
\]

Use \(A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\), \(B = \begin{bmatrix} 4 & 0 \\ -2 & -1 \end{bmatrix}\), \(C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\), and \(c = 3\) to determine whether the following equations are true for the given matrices.

15. \(AC = CA\)  
16. \(A(B + C) = BA + CA\)  
17. \((AB)c = c(AB)\)  
18. \((A + C)B = B(A + C)\)

19. **RENTALS** For their one-week vacation, the Montoyas can rent a 2-bedroom condominium for $1796, a 3-bedroom condominium for $2165, or a 4-bedroom condominium for $2538. The table shows the number of units in each of three complexes.

<table>
<thead>
<tr>
<th></th>
<th>2-Bedroom</th>
<th>3-Bedroom</th>
<th>4-Bedroom</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun Haven</td>
<td>36</td>
<td>24</td>
<td>22</td>
</tr>
<tr>
<td>Surfside</td>
<td>29</td>
<td>32</td>
<td>42</td>
</tr>
<tr>
<td>Seabreeze</td>
<td>18</td>
<td>22</td>
<td>18</td>
</tr>
</tbody>
</table>

**a.** Write a matrix that represents the number of each type of unit available at each complex and a matrix that represents the weekly charge for each type of unit.

**b.** If all of the units in the three complexes are rented for the week at the rates given the Montoyas, express the income of each of the three complexes as a matrix.

**c.** What is the total income of all three complexes for the week?
1. **FIND THE ERROR** Both \(A\) and \(B\) are 2 by 2 matrices. Maggie made the following derivation. Is this derivation valid? If not, what error did she make?
   
a. \((A + B)^2 = (A + B)(A + B)\)
b. \(= (A + B)A + (A + B)B\)
c. \(= AA + BA + AB + BB\)
d. \(= A^2 + BA + AB + B^2\)
e. \(= A^2 + AB + AB + B^2\)
f. \(= A^2 + 2AB + B^2\)

2. **EXAM SCORES** Mr. Farey recorded the exam scores of his students in a 20 by 3 matrix. Each row listed the scores of a different student. The first exam scores were listed in the first column, and the second exam scores were listed in the second column. The final exam scores were listed in the third column. Mr. Farey needed to create a 20 by 1 matrix that contained the weighted scores of each student. The first two exams account for 25% of the weighted score, and the final exam counted 50%. To make the matrix of weighted scores, what matrix can Mr. Farey multiply his 20 by 3 matrix by on the right?

3. **SPECIAL MATRICES** Mandy has a 3 by 3 matrix \(M\). She notices that for any 3 by 3 matrix \(X\), \(MX = X\). What must \(M\) be?

4. **POWERS** Thad just learned about matrix multiplication. He began to wonder what happens when you take powers of a matrix. He computed the first few powers of the matrix
   \[ M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \]
   and noticed a pattern. What is \(M^n\)?

5. **COST COMPARISONS** The average family spends more than $500 on school supplies at the beginning of each school year. Barbara and Lance need to buy pens, pencils, and erasers. They make a 2 by 3 matrix that represents the numbers of each item they would like to purchase.

<table>
<thead>
<tr>
<th></th>
<th>Pens</th>
<th>Pencils</th>
<th>Erasers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Barbara</td>
<td>10</td>
<td>15</td>
<td>3</td>
</tr>
<tr>
<td>Lance</td>
<td>5</td>
<td>20</td>
<td>5</td>
</tr>
</tbody>
</table>

   They call this matrix \(M\). Barbara and Lance find two stores that sell the items at different prices and record this information in a second matrix that they call \(P\).

<table>
<thead>
<tr>
<th></th>
<th>Store 1</th>
<th>Store 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pens</td>
<td>2.20</td>
<td>1.90</td>
</tr>
<tr>
<td>Pencils</td>
<td>0.85</td>
<td>0.95</td>
</tr>
<tr>
<td>Erasers</td>
<td>0.60</td>
<td>0.65</td>
</tr>
</tbody>
</table>

   a. Compute \(MP\).
   
b. What do the entries in \(MP\) mean?
4-4 Study Guide

Transformations with Matrices

Translations and Dilations Matrices that represent coordinates of points on a plane are useful in describing transformations.

- **Translation**: a transformation that moves a figure from one location to another on the coordinate plane

You can use matrix addition and a translation matrix to find the coordinates of the translated figure.

- **Dilation**: a transformation in which a figure is enlarged or reduced

You can use scalar multiplication to perform dilations.

**Example** Find the coordinates of the vertices of the image of ΔABC with vertices A(−5, 4), B(−1, 5), and C(−3, −1) if it is moved 6 units to the right and 4 units down. Then graph ΔABC and its image ΔA′B′C′.

Write the vertex matrix for ΔABC. \[
\begin{bmatrix}
-5 & -1 & -3 \\
4 & 5 & -1
\end{bmatrix}
\]

Add the translation matrix \[
\begin{bmatrix}
6 & 6 & 6 \\
-4 & -4 & -4
\end{bmatrix}
\]

to the vertex matrix of ΔABC.

\[
\begin{bmatrix}
-5 & -1 & -3 \\
4 & 5 & -1
\end{bmatrix}
+ \begin{bmatrix}
6 & 6 & 6 \\
-4 & -4 & -4
\end{bmatrix} = \begin{bmatrix}
1 & 5 & 3 \\
0 & 1 & -5
\end{bmatrix}
\]

The coordinates of the vertices of ΔA′B′C′ are A′(1, 0), B′(5, 1), and C′(3, −5).

**Exercises**

1. Quadrilateral QUAD with vertices Q(−1, −3), U(0, 0), A(5, −1), and D(2, −5) is translated 3 units to the left and 2 units up.
   - a. Write the translation matrix.
   - b. Find the coordinates of the vertices of Q′U′A′D′.

2. The vertices of ΔABC are A(4, −2), B(2, 8), and C(8, 2). The triangle is dilated so that its perimeter is one-fourth the original perimeter.
   - a. Write the coordinates of the vertices of ΔABC in a vertex matrix.
   - b. Find the coordinates of the vertices of image ΔA′B′C′.
   - c. Graph the preimage and the image.
Reflections and Rotations

<table>
<thead>
<tr>
<th>Reflection Matrices</th>
<th>For a reflection in the:</th>
<th>x-axis</th>
<th>y-axis</th>
<th>line y = x</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>multiply the vertex matrix on the left by:</td>
<td>[ \begin{pmatrix} 1 &amp; 0 \ 0 &amp; -1 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} -1 &amp; 0 \ 0 &amp; 1 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{pmatrix} ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rotation Matrices</th>
<th>For a counterclockwise rotation about the origin of:</th>
<th>90°</th>
<th>180°</th>
<th>270°</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>multiply the vertex matrix on the left by:</td>
<td>[ \begin{pmatrix} 0 &amp; 1 \ -1 &amp; 0 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} -1 &amp; 0 \ 0 &amp; -1 \end{pmatrix} ]</td>
<td>[ \begin{pmatrix} 0 &amp; 1 \ -1 &amp; 0 \end{pmatrix} ]</td>
</tr>
</tbody>
</table>

**Example**

Find the coordinates of the vertices of the image of \( \triangle ABC \) with \( A(3, 5), B(-2, 4), \) and \( C(1, -1) \) after a reflection in the line \( y = x \).

Write the ordered pairs as a vertex matrix. Then multiply the vertex matrix by the reflection matrix for \( y = x \).

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3 & -2 & 1 \\ 5 & 4 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 4 & -1 \\ 3 & -2 & -1 \end{pmatrix}
\]

The coordinates of the vertices of \( A'B'C' \) are \( A'(5, 3), B'(4, -2), \) and \( C'(-1, 1) \).

**Exercises**

1. The coordinates of the vertices of quadrilateral \( ABCD \) are \( A(-2, 1), B(-1, 3), C(2, 2), \) and \( D(2, -1) \). Find the coordinates of the vertices of the image \( A'B'C'D' \) after a reflection in the \( y \)-axis.

2. \( \triangle DEF \) with vertices \( D(-2, 5), E(1, 4), \) and \( F(0, -1) \) is rotated 90° counterclockwise about the origin.

   a. Write the coordinates of the triangle in a vertex matrix.

   b. Write the rotation matrix for this situation.

   c. Find the coordinates of the vertices of \( \triangle D'E'F' \).

   d. Graph the preimage and the image.
4-4 Practice

Transformations with Matrices

1. Quadrilateral $WXYZ$ with vertices $W(-3, 2)$, $X(-2, 4)$, $Y(4, 1)$, and $Z(3, 0)$ is translated 1 unit left and 3 units down.

   a. Write the translation matrix.

   b. Find the coordinates of quadrilateral $W'X'Y'Z'$.

   c. Graph the preimage and the image.

2. The vertices of $\triangle RST$ are $R(6, 2)$, $S(3, -3)$, and $T(-2, 5)$. The triangle is dilated so that its perimeter is one half the original perimeter.

   a. Write the coordinates of $\triangle RST$ in a vertex matrix.

   b. Find the coordinates of the image $\triangle R'S'T'$.

   c. Graph $\triangle RST$ and $\triangle R'S'T'$.

3. The vertices of quadrilateral $ABCD$ are $A(-3, 2)$, $B(0, 3)$, $C(4, -4)$, and $D(-2, -2)$. The quadrilateral is reflected in the $y$-axis.

   a. Write the coordinates of $ABCD$ in a vertex matrix.

   b. Write the reflection matrix for this situation.

   c. Find the coordinates of $A'B'C'D'$.

   d. Graph $ABCD$ and $A'B'C'D'$.

4. ARCHITECTURE Using architectural design software, the Bradleys plot their kitchen plans on a grid with each unit representing 1 foot. They place the corners of an island at $(2, 8)$, $(8, 11)$, $(3, 5)$, and $(9, 8)$. If the Bradleys wish to move the island 1.5 feet to the right and 2 feet down, what will the new coordinates of its corners be?

5. BUSINESS The design of a business logo calls for locating the vertices of a triangle at $(1.5, 5)$, $(4, 1)$, and $(1, 0)$ on a grid. If design changes require rotating the triangle $90^\circ$ counterclockwise, what will the new coordinates of the vertices be?
1. **ICONS** Louis needs to perform many matrix transformations to the basic house icon shown in the graph.

What is the vertex matrix for this image?

2. **LANDSCAPING** Using the center as the origin, a landscaper placed features at the given coordinates in the northern half of the Great Lawn of Grand Central Park in New York City: a fountain (75, 200), a rock sculpture (150, 175), a bench (−150, 130), and a plaque (0, 260). Use a reflection matrix to find the coordinates for features the landscaper can place so that the south half will be a reflection of the north.

3. **MIRROR SYMMETRY** A detective found only half of an image with mirror symmetry about the line \( y = x \). The vertex matrix of the visible part is
\[
\begin{bmatrix}
4 & 5 & -2 \\
2 & -5 & -4
\end{bmatrix}
\]
What are the coordinates of the hidden vertices?

4. **PHOTOGRAPHY** Alejandra used a digital camera to take a picture. Because she held the camera sideways, the image on her computer screen appeared sideways. In order to transform the picture, she needed to perform a 90° clockwise rotation. What matrix represents this transformation?

5. **ARROWS** A compass arrow is pointing Northeast.

a. What is the vertex matrix for the arrow?

b. What would the vertex matrix be for the arrow if it were pointing Northwest? (Hint: Rotate 90° around the origin.)
Determinants and Cramer’s Rule

**Determinants** A $2 \times 2$ matrix has a second-order determinant; a $3 \times 3$ matrix has a third-order determinant.

### Second-Order Determinant

For the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

### Third-Order Determinant

For the matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, the determinant is found using the diagonal rule.

### Area of a Triangle

The area of a triangle having vertices $(a, b)$, $(c, d)$, and $(e, f)$ is $|A|$, where $A = \frac{1}{2} \begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$.

### Example

**Evaluate each determinant.**

**a.** $\begin{vmatrix} 6 & 3 \\ -8 & 5 \end{vmatrix}$

$\begin{vmatrix} 6 & 3 \\ -8 & 5 \end{vmatrix} = 6(5) - 3(-8) = 54$

**b.** $\begin{vmatrix} 4 & 5 & -2 \\ 1 & 3 & 0 \\ 2 & -3 & 6 \end{vmatrix}$

$\begin{vmatrix} 4 & 5 & -2 \\ 1 & 3 & 0 \\ 2 & -3 & 6 \end{vmatrix} = [4(3)6 + 5(0)2 + (-2)1(-3)] - [(-2)3(2) + 4(0)(-3) + 5(1)6] = [72 + 0 + 6] - [-12 + 0 + 30] = 78 - 16 = 60$

**Exercises**

Evaluate each determinant.

1. $\begin{vmatrix} 6 & -2 \\ 5 & 7 \end{vmatrix}$

2. $\begin{vmatrix} 3 & 2 \\ 9 & 6 \end{vmatrix}$

3. $\begin{vmatrix} 3 & -2 & -2 \\ 0 & 4 & 1 \\ -1 & 4 & -3 \end{vmatrix}$

4. Find the area of a triangle with vertices $(2, -3)$, $(7, 4)$, and $(-5, 5)$. 

**Exercise Solution**

The area of a triangle with vertices $(a, b)$, $(c, d)$, and $(e, f)$ is $|A|$, where $A = \frac{1}{2} \begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$. For vertices $(2, -3)$, $(7, 4)$, and $(-5, 5)$, we have

$A = \frac{1}{2} \begin{vmatrix} 2 & -3 & 1 \\ 7 & 4 & 1 \\ -5 & 5 & 1 \end{vmatrix}$

$A = \frac{1}{2} [2(4)(1) + (-3)(1)(1) + (1)(7)(5) - 1(7)(1) - 2(-3)(1) - 1(-5)(4)]$

$A = \frac{1}{2} [8 - 3 + 35 - 7 + 6 + 20]$

$A = \frac{1}{2} [51]$

$A = \frac{51}{2}$
Cramer’s Rule  Determinants provide a way for solving systems of equations.

Let \( C \) be the coefficient matrix of the system

\[
\begin{align*}
ax + by &= m \\
fx + gy &= n
\end{align*}
\]

The solution of this system is

\[
\begin{align*}
x &= \frac{\begin{vmatrix} m & b \\ n & g \end{vmatrix}}{|C|}, \\
y &= \frac{\begin{vmatrix} a & m \\ f & n \end{vmatrix}}{|C|}, \quad \text{if } C \neq 0.
\end{align*}
\]

Example  Use Cramer’s Rule to solve the system of equations.

\[
\begin{align*}
5x - 10y &= 8 \\
10x + 25y &= -2
\end{align*}
\]

\[
x = \frac{\begin{vmatrix} 8 & -10 \\ -2 & 25 \end{vmatrix}}{|C|}, \quad y = \frac{\begin{vmatrix} 5 & 8 \\ 10 & -2 \end{vmatrix}}{|C|}
\]

\[
\begin{align*}
x &= \frac{8(25) - (-2)(-10)}{5(25) - (-10)(10)} \\
y &= \frac{5(-2) - 8(10)}{5(25) - (-10)(10)}
\end{align*}
\]

\[
\begin{align*}
x &= \frac{180}{225}, \quad y = \frac{-90}{225}
\end{align*}
\]

The solution is \((\frac{4}{5}, \frac{-2}{5})\).

Exercises

Use Cramer’s Rule to solve each system of equations.

1. \(3x - 2y = 7\)  \(2x + 7y = 38\)
2. \(x - 4y = 17\)  \(3x - y = 29\)
3. \(2x - y = -2\)  \(4x - y = 4\)
4. \(2x - y = 1\)  \(5x + 2y = -29\)
5. \(4x + 2y = 1\)  \(5x - 4y = 24\)
6. \(6x - 3y = -3\)  \(2x + y = 21\)
7. \(2x + 7y = 16\)  \(x - 2y = 30\)
8. \(2x - 3y = -2\)  \(3x - 4y = 9\)
9. \(\frac{x}{3} + \frac{y}{5} = 2\)  \(\frac{x}{4} - \frac{y}{6} = -8\)
10. \(6x - 9y = -1\)  \(3x + 18y = 12\)
11. \(3x - 12y = -14\)  \(9x + 6y = -7\)
12. \(8x + 2y = \frac{3}{7}\)  \(5x - 4y = -\frac{27}{7}\)
4-5 Practice

Determinants and Cramer's Rule

Evaluate each determinant.

1. \[ \begin{array}{cc} 1 & 6 \\ 2 & 7 \end{array} \]
2. \[ \begin{array}{cc} 9 & 6 \\ 3 & 2 \end{array} \]
3. \[ \begin{array}{cc} 4 & 1 \\ -2 & -5 \end{array} \]
4. \[ \begin{array}{cc} -14 & -3 \\ 2 & -2 \end{array} \]
5. \[ \begin{array}{cc} 4 & -3 \\ -12 & 4 \end{array} \]
6. \[ \begin{array}{cc} 2 & -5 \\ 5 & -11 \end{array} \]
7. \[ \begin{array}{cc} 3 & -4 \\ 3.75 & 5 \end{array} \]
8. \[ \begin{array}{cc} 2 & -1 \\ 3 & -9.5 \end{array} \]
9. \[ \begin{array}{cc} 0.5 & -0.7 \\ 0.4 & -0.3 \end{array} \]

Evaluate each determinant using expansion by diagonals.

10. \[ \begin{array}{ccc} -2 & 3 & 1 \\ 0 & 4 & -3 \\ 2 & 5 & -1 \end{array} \]
11. \[ \begin{array}{ccc} 2 & -4 & 1 \\ 3 & 0 & 9 \\ -1 & 5 & 7 \end{array} \]
12. \[ \begin{array}{ccc} 2 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & -1 \end{array} \]
13. \[ \begin{array}{ccc} 0 & -4 & 0 \\ 2 & -1 & 1 \\ 3 & -2 & 5 \end{array} \]
14. \[ \begin{array}{ccc} 2 & 7 & -6 \\ 8 & 4 & 0 \\ 1 & -1 & 3 \end{array} \]
15. \[ \begin{array}{ccc} -12 & 0 & 3 \\ 7 & 5 & -1 \\ 4 & 2 & -6 \end{array} \]

Use Cramer's Rule to solve each system of equation.

16. \[ 4x - 2y = -6 \]
\[ 3x + y = 18 \]
17. \[ 5x + 4y = 10 \]
\[ -3x - 2y = -8 \]
18. \[ -2x - 3y = -14 \]
\[ 4x - y = 0 \]
19. \[ 6x + 6y = 9 \]
\[ 4x - 4y = -42 \]
20. \[ 5x - 6 = 3y \]
\[ 5y = 54 + 3x \]
21. \[ \frac{x}{2} + \frac{y}{4} = 2 \]
\[ \frac{x}{4} - \frac{y}{6} = -6 \]

25. GEOMETRY Find the area of a triangle whose vertices have coordinates (3, 5), (6, -5), and (-4, 10).

26. LAND MANAGEMENT A fish and wildlife management organization uses a GIS (geographic information system) to store and analyze data for the parcels of land it manages. All of the parcels are mapped on a grid in which 1 unit represents 1 acre. If the coordinates of the corners of a parcel are (-8, 10), (6, 17), and (2, -4), how many acres is the parcel?
4-5 Word Problem Practice

**Determinants and Cramer’s Rule**

1. **FIND THE ERROR** Mark’s determinant computation has sign errors. Circle the signs that must be reversed.

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{vmatrix} = 1(5)(9) - 2(6)(7) + 3(4)(8) - 3(5)(7) + 1(6)(8) - 2(4)(9)
\]

2. **POOL** An architect has a pool in the floor plans for a home. Set up a determinant that gives the unit area of the pool.

3. **HALF-UNIT TRIANGLES** For a school art project, students had to decorate a pegboard by looping strings around the pegs. Ronald wanted to make triangles with areas of one half square unit. Because Ronald had studied determinants, he knew that this was essentially the same as finding the coordinates of the vertices of a triangle \((a, b), (c, d)\) and \((e, f)\), so that the determinant

\[
\begin{vmatrix}
a & b & 1 \\
c & d & 1 \\
e & f & 1 \\
\end{vmatrix}
\]

is 1 or -1.

Give an example of such a triangle.

4. **ITALY** The figure shows a map of Italy overlaid on a graph. The coordinates of Milan, Venice, and Pisa are about \((-4, 5), (3.25, 4.8), \) and \((-1.4, -0.8), \) respectively. Each square unit on the map represents about 400 square miles.

What is the area of the triangular region? Round your answer to the nearest square mile.

5. **ARROWS** Kyle is making a triangle with vertices at \((-6, 0), (0, -x), \) and \((0, x), \) and \(x > 0.\) He plans to make the triangle using a material that costs $2 for every square unit.

   **a.** Write the determinant that gives the area of this triangle.

   **b.** Evaluate the determinant you wrote for part **a** and determine the value of \(x\) that results in a $60 triangle.
Inverse Matrices and Systems of Equations

Identity and Inverse Matrices The identity matrix for matrix multiplication is a square matrix with 1s for every element of the main diagonal and zeros elsewhere.

| Identity Matrix for Multiplication | If $A$ is an $n \times n$ matrix and $I$ is the identity matrix, then $A \cdot I = A$ and $I \cdot A = A$. |

If an $n \times n$ matrix $A$ has an inverse $A^{-1}$, then $A \cdot A^{-1} = A^{-1} \cdot A = I$.

Example

Determine whether $X = \begin{bmatrix} 7 & 4 \\ 10 & 6 \end{bmatrix}$ and $Y = \begin{bmatrix} 3 & -2 \\ -5 & 7/2 \end{bmatrix}$ are inverse matrices.

Find $X \cdot Y$.

$X \cdot Y = \begin{bmatrix} 7 & 4 \\ 10 & 6 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ -5 & 7/2 \end{bmatrix}$

$= \begin{bmatrix} 21 - 20 & -14 + 14 \\ 30 - 30 & -20 + 21 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Find $Y \cdot X$.

$Y \cdot X = \begin{bmatrix} 3 & -2 \\ -5 & 7/2 \end{bmatrix} \cdot \begin{bmatrix} 7 & 4 \\ 10 & 6 \end{bmatrix}$

$= \begin{bmatrix} 21 - 20 & 12 - 12 \\ -35 + 35 & -20 + 21 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Since $X \cdot Y = Y \cdot X = I$, $X$ and $Y$ are inverse matrices.

Exercises

Determine whether the matrices in each pair are inverses of each other.

1. $\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ -5/2 & 3/2 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$

4. $\begin{bmatrix} 8 & 11 \\ 3 & 14 \end{bmatrix}$ and $\begin{bmatrix} -4 & 11 \\ 3 & -8 \end{bmatrix}$

5. $\begin{bmatrix} 4 & -1 \\ 5 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$

6. $\begin{bmatrix} 5 & 2 \\ 11 & 4 \end{bmatrix}$ and $\begin{bmatrix} -2 & 1 \\ 11/2 & -5/2 \end{bmatrix}$

7. $\begin{bmatrix} 4 & 2 \\ 6 & -2 \end{bmatrix}$ and $\begin{bmatrix} -1 & -1 \\ 5/10 & -10 \end{bmatrix}$

8. $\begin{bmatrix} 5 & 8 \\ 4 & 6 \end{bmatrix}$ and $\begin{bmatrix} -3 & 4 \\ 2 & -5/2 \end{bmatrix}$

9. $\begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}$ and $\begin{bmatrix} 7/2 & -3/2 \\ 1 & -2 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 2 \\ 4 & -6 \end{bmatrix}$ and $\begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$

11. $\begin{bmatrix} 7 & 2 \\ 17 & 5 \end{bmatrix}$ and $\begin{bmatrix} 5 & -2 \\ -17 & 7 \end{bmatrix}$

12. $\begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix}$ and $\begin{bmatrix} -5 & 3 \\ 7 & -4 \end{bmatrix}$
Inverse Matrices and Systems of Equations

Matrix Equations A matrix equation for a system of equations consists of the product of the coefficient and variable matrices on the left and the constant matrix on the right of the equals sign.

Example Use a matrix equation to solve a system of equations.

\[
\begin{align*}
3x - 7y &= 12 \\
x + 5y &= -8
\end{align*}
\]

Determine the coefficient, variable, and constant matrices.

\[
\begin{bmatrix}
3 & -7 \\
1 & 5
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
12 \\
-8
\end{bmatrix}
\]

Find the inverse of the coefficient matrix.

\[
\frac{1}{3(5) - 1(-7)}
\begin{bmatrix}
5 & 7 \\
-1 & 3
\end{bmatrix}
= 
\frac{1}{22}
\begin{bmatrix}
7 & 22 \\
-1 & 3
\end{bmatrix}
\]

Rewrite the equation in the form of \(X = A^{-1}B\)

\[
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
5 & 7 \\
-1 & 3
\end{bmatrix}
\begin{bmatrix}
12 \\
-8
\end{bmatrix}
\]

Solve.

\[
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
\frac{2}{11} \\
-\frac{18}{11}
\end{bmatrix}
\]

Exercises

Use a matrix equation to solve each system of equations.

1. \(2x + y = 8\) \hspace{1cm} 2. \(4x - 3y = 18\)
   \(5x - 3y = -12\) \hspace{1cm} \(x + 2y = 12\)

3. \(7x - 2y = 15\) \hspace{1cm} 4. \(4x - 6y = 20\)
   \(3x + y = -10\) \hspace{1cm} \(3x + y + 8 = 0\)

5. \(5x + 2y = 18\) \hspace{1cm} 6. \(3x - y = 24\)
   \(x = -4y + 25\) \hspace{1cm} \(3y = 80 - 2x\)
4-6 Practice

**Inverse Matrices and Systems of Equations**

Determine whether each pair of matrices are inverses.

1. \( M = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, N = \begin{bmatrix} -2 & 1 \\ 3 & -2 \end{bmatrix} \)

2. \( X = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}, Y = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \)

3. \( A = \begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{5} & -\frac{1}{10} \\ \frac{2}{5} & \frac{3}{10} \end{bmatrix} \)

4. \( P = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}, Q = \begin{bmatrix} \frac{3}{14} & \frac{1}{7} \\ \frac{1}{7} & \frac{3}{7} \end{bmatrix} \)

Determine whether each statement is true or false.

5. All square matrices have multiplicative inverses.

6. All square matrices have multiplicative identities.

Find the inverse of each matrix, if it exists.

7. \( \begin{bmatrix} 4 & 5 \\ -4 & -3 \end{bmatrix} \)

8. \( \begin{bmatrix} 2 & 0 \\ 3 & 5 \end{bmatrix} \)

9. \( \begin{bmatrix} -1 & 3 \\ 4 & -7 \end{bmatrix} \)

10. \( \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix} \)

11. \( \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix} \)

12. \( \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \)

13. **GEOMETRY** Use the figure at the right.

   a. Write the vertex matrix \( A \) for the rectangle.

   b. Use matrix multiplication to find \( BA \) if \( B = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} \).

   c. Graph the vertices of the transformed quadrilateral on the previous graph.

   d. Make a conjecture about what transformation \( B^{-1} \) describes on a coordinate plane.

14. **CODES** Use the alphabet table below and the inverse of coding matrix \( C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \) to decode this message:

\[
\begin{array}{cccccccccccc}
19 & 14 & 11 & 13 & 11 & 22 & 55 & 65 & 57 & 60 & 2 & 1 & 52 & 47 & 33 & 51 & 56 & 55
\end{array}
\]

<table>
<thead>
<tr>
<th>CODE</th>
</tr>
</thead>
<tbody>
<tr>
<td>A 1</td>
</tr>
<tr>
<td>B 2</td>
</tr>
<tr>
<td>C 3</td>
</tr>
<tr>
<td>D 4</td>
</tr>
<tr>
<td>E 5</td>
</tr>
<tr>
<td>F 6</td>
</tr>
<tr>
<td>G 7</td>
</tr>
<tr>
<td>H 8</td>
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<td>J 10</td>
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<td>K 11</td>
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</tr>
<tr>
<td>Q 17</td>
</tr>
<tr>
<td>R 18</td>
</tr>
<tr>
<td>S 19</td>
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<tr>
<td>T 20</td>
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<tr>
<td>U 21</td>
</tr>
<tr>
<td>V 22</td>
</tr>
<tr>
<td>W 23</td>
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<tr>
<td>X 24</td>
</tr>
<tr>
<td>Y 25</td>
</tr>
<tr>
<td>Z 26</td>
</tr>
<tr>
<td>- 0</td>
</tr>
</tbody>
</table>

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Chapter 4  55  North Carolina StudyText, Math BC, Volume 2
1. **TEACHING** Paula is explaining matrices to her father. She writes down the following system of equations.

   \[2x + y = 4\]
   \[3x + y = 5.\]

   Next, Paula shows her father the matrices that correspond to this system of equations. What are the matrices?

2. **TRANSPORTATION** Paula wrote the following matrix equation to show the costs of two trips by water taxi to Logan Airport in Boston. She used \(x\) for the cost of round trips and \(y\) for the cost one one-way trips.

   \[
   \begin{bmatrix}
   3 & 1 \\
   2 & 2
   \end{bmatrix}
   \begin{bmatrix}
   x \\
   y
   \end{bmatrix} =
   \begin{bmatrix}
   61 \\
   54
   \end{bmatrix}
   \]

   Next, she found the inverse.

   \[
   \begin{bmatrix}
   3 & 1 \\
   2 & 2
   \end{bmatrix}^{-1} = \frac{1}{4}
   \begin{bmatrix}
   2 & -1 \\
   -2 & 3
   \end{bmatrix}
   \]

   Then she computed her answer.

   \[
   \begin{bmatrix}
   x \\
   y
   \end{bmatrix} =
   \begin{bmatrix}
   \frac{1}{2} & -\frac{1}{4} \\
   -\frac{1}{2} & \frac{3}{4}
   \end{bmatrix}
   \begin{bmatrix}
   61 \\
   54
   \end{bmatrix} =
   \begin{bmatrix}
   34 \\
   10
   \end{bmatrix}
   \]

   When she checked her answer, the total cost of the trips came out as $112 and $88. Where did she make a mistake?

3. **AGES** Hank, Laura, and Ned are ages \(h\), \(l\), and \(n\), respectively. The sum of their ages is 15 years. Laura is one year younger than the sum of Hank and Ned’s ages. Ned is three times as old as Hank. Use matrices to determine the age of each sibling.

4. **SELF-INVERSES** Phillip notices that any matrix with ones and negative ones on the diagonal and zeroes everywhere else has the property that it is its own inverse. Give an example of a 2 by 2 matrix that is its own inverse but has at least 1 nonzero number off the diagonal.

5. **MATRIX OPERATIONS** Garth is studying determinants and inverses of matrices in math class. His teacher suggests that there are some matrices with unique properties, and challenges the class to find such matrices and describe the properties found. Garth is curious about the matrix \(G = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\).

   a. What is the determinant of \(G\)?

   b. Does the inverse of \(G\) exist? Explain.

   c. Determine a matrix operation that could be used to transform \(G\) into its Additive Identity matrix.
Graphing Quadratic Functions

Graph Quadratic Functions

<table>
<thead>
<tr>
<th>Quadratic Function</th>
<th>a function defined by an equation of the form ( f(x) = ax^2 + bx + c ), where ( a \neq 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph of a Quadratic Function</td>
<td>a parabola with these characteristics: y-intercept: ( c ); axis of symmetry: ( x = \frac{-b}{2a} ); x-coordinate of vertex: ( \frac{-b}{2a} )</td>
</tr>
</tbody>
</table>

Example

Find the y-intercept, the equation of the axis of symmetry, and the x-coordinate of the vertex for the graph of \( f(x) = x^2 - 3x + 5 \). Use this information to graph the function.

\( a = 1, b = -3, \) and \( c = 5 \), so the y-intercept is 5. The equation of the axis of symmetry is \( x = \frac{-(-3)}{2(1)} \) or \( \frac{3}{2} \). The x-coordinate of the vertex is \( \frac{3}{2} \).

Next make a table of values for \( x \) near \( \frac{3}{2} \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x^2 - 3x + 5 )</th>
<th>( f(x) )</th>
<th>( (x, f(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0^2 - 3(0) + 5</td>
<td>5</td>
<td>(0, 5)</td>
</tr>
<tr>
<td>1</td>
<td>1^2 - 3(1) + 5</td>
<td>3</td>
<td>(1, 3)</td>
</tr>
<tr>
<td>( \frac{3}{2} )</td>
<td>( \left( \frac{3}{2} \right)^2 - 3\left( \frac{3}{2} \right) + 5 )</td>
<td>( \frac{11}{4} )</td>
<td>( \left( \frac{3}{2}, \frac{11}{4} \right) )</td>
</tr>
<tr>
<td>2</td>
<td>2^2 - 3(2) + 5</td>
<td>3</td>
<td>(2, 3)</td>
</tr>
<tr>
<td>3</td>
<td>3^2 - 3(3) + 5</td>
<td>5</td>
<td>(3, 5)</td>
</tr>
</tbody>
</table>

Exercises

Complete parts a–c for each quadratic function.

a. Find the y-intercept, the equation of the axis of symmetry, and the x-coordinate of the vertex.

b. Make a table of values that includes the vertex.

c. Use this information to graph the function.

1. \( f(x) = x^2 + 6x + 8 \)
2. \( f(x) = -x^2 - 2x + 2 \)
3. \( f(x) = 2x^2 - 4x + 3 \)
Graphing Quadratic Functions

Maximum and Minimum Values The y-coordinate of the vertex of a quadratic function is the **maximum value** or **minimum value** of the function.

| Maximum or Minimum Value of a Quadratic Function | The graph of $f(x) = ax^2 + bx + c$, where $a \neq 0$, opens up and has a minimum when $a > 0$. The graph opens down and has a maximum when $a < 0$. |

### Example
Determine whether each function has a **maximum** or **minimum** value, and find that value. Then state the domain and range of the function.

#### a. $f(x) = 3x^2 - 6x + 7$
- For this function, $a = 3$ and $b = -6$.
- Since $a > 0$, the graph opens up, and the function has a minimum value.
- The minimum value is the y-coordinate of the vertex. The x-coordinate of the vertex is $\frac{-b}{2a} = \frac{-(-6)}{2(3)} = 1$.
- Evaluate the function at $x = 1$ to find the minimum value.
- $f(1) = 3(1)^2 - 6(1) + 7 = 4$, so the minimum value of the function is 4. The domain is all real numbers. The range is all reals greater than or equal to the minimum value, that is $\{f(x) \mid f(x) \geq 4\}$.

#### b. $f(x) = 100 - 2x - x^2$
- For this function, $a = -1$ and $b = -2$.
- Since $a < 0$, the graph opens down, and the function has a maximum value.
- The maximum value is the y-coordinate of the vertex. The x-coordinate of the vertex is $\frac{-b}{2a} = \frac{-(-2)}{2(-1)} = -1$.
- Evaluate the function at $x = -1$ to find the maximum value.
- $f(-1) = 100 - 2(-1) - (-1)^2 = 101$, so the maximum value of the function is 101. The domain is all real numbers. The range is all reals less than or equal to the maximum value, that is $\{f(x) \mid f(x) \leq 101\}$.

### Exercises
Determine whether each function has a **maximum** or **minimum** value, and find that value. Then state the domain and range of the function.

1. $f(x) = 2x^2 - x + 10$
2. $f(x) = x^2 + 4x - 7$
3. $f(x) = 3x^2 - 3x + 1$
4. $f(x) = x^2 + 5x + 2$
5. $f(x) = 20 + 6x - x^2$
6. $f(x) = 4x^2 + x + 3$
7. $f(x) = -x^2 - 4x + 10$
8. $f(x) = x^2 - 10x + 5$
9. $f(x) = -6x^2 + 12x + 21$
### 5-1 Practice

**Graphing Quadratic Functions**

Complete parts a–c for each quadratic function.

**a.** Find the $y$-intercept, the equation of the axis of symmetry, and the $x$-coordinate of the vertex.

**b.** Make a table of values that includes the vertex.

**c.** Use this information to graph the function.

1. $f(x) = x^2 - 8x + 15$
2. $f(x) = -x^2 - 4x + 12$
3. $f(x) = 2x^2 - 2x + 1$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>10</td>
</tr>
<tr>
<td>0</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
</tbody>
</table>

Determine whether each function has a **maximum** or **minimum** value, and find that value. Then state the domain and range of the function.

4. $f(x) = x^2 + 2x - 8$
5. $f(x) = x^2 - 6x + 14$
6. $v(x) = -x^2 + 14x - 57$

7. $f(x) = 2x^2 + 4x - 6$
8. $f(x) = -x^2 + 4x - 1$
9. $f(x) = -\frac{2}{3}x^2 + 8x - 24$

10. **GRAVITATION** From 4 feet above a swimming pool, Susan throws a ball upward with a velocity of 32 feet per second. The height $h(t)$ of the ball $t$ seconds after Susan throws it is given by $h(t) = -16t^2 + 32t + 4$. For $t \geq 0$, find the maximum height reached by the ball and the time that this height is reached.

11. **HEALTH CLUBS** Last year, the SportsTime Athletic Club charged $20 to participate in an aerobics class. Seventy people attended the classes. The club wants to increase the class price this year. They expect to lose one customer for each $1 increase in the price.

   **a.** What price should the club charge to maximize the income from the aerobics classes?

   **b.** What is the maximum income the SportsTime Athletic Club can expect to make?
1. **TRAJECTORIES** A cannonball is launched from a cannon on the wall of Fort Chambly, Quebec. If the path of the cannonball is traced on a piece of graph paper aligned so that the cannon is situated on the y-axis, the equation that describes the path is

\[ y = -\frac{1}{1600}x^2 + \frac{1}{2}x + 20, \]

where \( x \) is the horizontal distance from the cliff and \( y \) is the vertical distance above the ground in feet. How high above the ground is the cannon?

2. **TICKETING** The manager of a symphony computes that the symphony will earn \(-40P^2 + 1100P\) dollars per concert if they charge \( P \) dollars for tickets. What ticket price should the symphony charge in order to maximize its profits?

3. **ARCHES** An architect decides to use a parabolic arch for the main entrance of a science museum. In one of his plans, the top edge of the arch is described by the graph of \( y = -\frac{1}{4}x^2 + \frac{5}{2}x + 15 \). What are the coordinates of the vertex of this parabola?

4. **FRAMING** A frame company offers a line of square frames. If the side length of the frame is \( s \), then the area of the opening in the frame is given by the function \( a(s) = s^2 - 10s + 24 \). Graph \( a(s) \).

5. **WALKING** Canal Street and Walker Street are perpendicular to each other. Evita is driving south on Canal Street and is currently 5 miles north of the intersection with Walker Street. Jack is at the intersection of Canal and Walker Streets and heading east on Walker. Jack and Evita are both driving 30 miles per hour.

   a. When Jack is \( x \) miles east of the intersection, where is Evita?

   b. The distance between Jack and Evita is given by the formula \( \sqrt{x^2 + (5 - x)^2} \). For what value of \( x \) are Jack and Evita at their closest? (Hint: Minimize the square of the distance.)

   c. What is the distance of closest approach?
Solving Quadratic Equations by Graphing

A quadratic equation has the form $ax^2 + bx + c = 0$, where $a \neq 0$.

The zeros of a quadratic function are the $x$-intercepts of its graph. Therefore, finding the $x$-intercepts is one way of solving the related quadratic equation.

**Example**

Solve $x^2 + x - 6 = 0$ by graphing.

Graph the related function $f(x) = x^2 + x - 6$.

The $x$-coordinate of the vertex is $\frac{-b}{2a} = \frac{-1}{2}$, and the equation of the axis of symmetry is $x = \frac{-1}{2}$.

Make a table of values using $x$-values around $\frac{-1}{2}$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$-6$</td>
</tr>
<tr>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{1}{4}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$-6$</td>
</tr>
<tr>
<td>$1$</td>
<td>$-4$</td>
</tr>
<tr>
<td>$2$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

From the table and the graph, we can see that the zeros of the function are 2 and $-3$.

**Exercises**

Use the related graph of each equation to determine its solution.

1. $x^2 + 2x - 8 = 0$
2. $x^2 - 4x - 5 = 0$
3. $x^2 - 5x + 4 = 0$
4. $x^2 - 10x + 21 = 0$
5. $x^2 + 4x + 6 = 0$
6. $4x^2 + 4x + 1 = 0$
Solving Quadratic Equations by Graphing

Estimate Solutions  Often, you may not be able to find exact solutions to quadratic equations by graphing. But you can use the graph to estimate solutions.

**Example**  Solve $x^2 - 2x - 2 = 0$ by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located.

The equation of the axis of symmetry of the related function is $x = \frac{-(-2)}{2(1)} = 1$, so the vertex has $x$-coordinate 1. Make a table of values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1</td>
<td>-2</td>
<td>-3</td>
<td>-2</td>
<td>1</td>
</tr>
</tbody>
</table>

The $x$-intercepts of the graph are between 2 and 3 and between 0 and $-1$. So one solution is between 2 and 3, and the other solution is between 0 and $-1$.

**Exercises**

Solve the equations. If exact roots cannot be found, state the consecutive integers between which the roots are located.

1. $x^2 - 4x + 2 = 0$
2. $x^2 + 6x + 6 = 0$
3. $x^2 + 4x + 2 = 0$
4. $-x^2 + 2x + 4 = 0$
5. $2x^2 - 12x + 17 = 0$
6. $-\frac{1}{2}x^2 + x + \frac{5}{2} = 0$
5-2 Practice

Solving Quadratic Equations By Graphing

Use the related graph of each equation to determine its solutions.

1. \(-3x^2 + 3 = 0\)
2. \(3x^2 + x + 3 = 0\)
3. \(x^2 - 3x + 2 = 0\)

Solve each equation. If exact roots cannot be found, state the consecutive integers between which the roots are located.

4. \(-2x^2 - 6x + 5 = 0\)
5. \(x^2 + 10x + 24 = 0\)
6. \(2x^2 - x - 6 = 0\)

7. \(-x^2 + x + 6 = 0\)
8. \(-x^2 + 5x - 8 = 0\)

9. GRAVITY Use the formula \(h(t) = v_0 t - 16t^2\), where \(h(t)\) is the height of an object in feet, \(v_0\) is the object’s initial velocity in feet per second, and \(t\) is the time in seconds.

a. Marta throws a baseball with an initial upward velocity of 60 feet per second. Ignoring Marta’s height, how long after she releases the ball will it hit the ground?

b. A volcanic eruption blasts a boulder upward with an initial velocity of 240 feet per second. How long will it take the boulder to hit the ground if it lands at the same elevation from which it was ejected?
5-2 Word Problem Practice

Solving Quadratic Equations by Graphing

1. **TRAJECTORIES** David threw a baseball into the air. The function of the height of the baseball in feet is \( h = 80t - 16t^2 \), where \( t \) represents the time in seconds after the ball was thrown. Use this graph of the function to determine how long it took for the ball to fall back to the ground.

![Graph of Trajectories](image)

2. **BRIDGES** In 1895, a brick arch railway bridge was built on North Avenue in Baltimore, Maryland. The arch is described by the equation \( h = 9 - \frac{1}{50}x^2 \), where \( h \) is the height in yards and \( x \) is the distance in yards from the center of the bridge. Graph this equation and describe, to the nearest yard, where the bridge touches the ground.

![Graph of Bridges](image)

3. **LOGIC** Wilma is thinking of two numbers. The sum is 2 and the product is -24. Use a quadratic equation to find the two numbers.

4. **RADIO TELESCOPES** The cross-section of a large radio telescope is a parabola. The dish is set into the ground. The equation that describes the cross-section is \( d = \frac{2}{75}x^2 - \frac{4}{3}x - \frac{32}{3} \), where \( d \) gives the depth of the dish below ground and \( x \) is the distance from the control center, both in meters. If the dish does not extend above the ground level, what is the diameter of the dish? Solve by graphing.

![Graph of Radio Telescopes](image)

5. **BOATS** The distance between two boats is

\[
d = \sqrt{t^2 - 10t + 35},
\]

where \( d \) is distance in meters and \( t \) is time in seconds.

a. Make a graph of \( d^2 \) versus \( t \).

![Graph of Boats](image)

b. Do the boats ever collide?
Factored Form  To write a quadratic equation with roots \( p \) and \( q \), let \((x - p)(x - q) = 0\). Then multiply using FOIL.

**Example**  Write a quadratic equation in standard form with the given roots.

a. 3, –5

\[
(x - p)(x - q) = 0 \quad \text{Write the pattern.}
\]

\[
(x - 3)(x - (-5)) = 0 \quad \text{Replace } p \text{ with } 3, q \text{ with } -5.
\]

\[
x^2 + 2x - 15 = 0 \quad \text{Use FOIL.}
\]

The equation \( x^2 + 2x - 15 = 0 \) has roots 3 and –5.

b. \(-\frac{7}{8}, \frac{1}{3}\)

\[
(x - p)(x - q) = 0
\]

\[
\left[ x - \left( -\frac{7}{8} \right) \right] \left( x - \frac{1}{3} \right) = 0
\]

\[
\frac{8}{3} \cdot \frac{24}{24} \cdot \frac{x + \frac{7}{8}}{3} \cdot \frac{3x - 1}{3} = 0
\]

\[
24 \cdot (8x + 7)(3x - 1) = 0
\]

\[
24x^2 + 13x - 7 = 0
\]

The equation \( 24x^2 + 13x - 7 = 0 \) has roots \(-\frac{7}{8}\) and \(\frac{1}{3}\).

**Exercises**

Write a quadratic equation in standard form with the given root(s).

1. 3, –4

2. –8, –2

3. 1, 9

4. –5

5. 10, 7

6. –2, 15

7. \(-\frac{1}{3}, 5\)

8. 2, \(\frac{2}{3}\)

9. \(-7, \frac{3}{4}\)

10. 3, \(\frac{2}{5}\)

11. \(-\frac{4}{9}, -1\)

12. 9, \(\frac{1}{6}\)

13. \(\frac{2}{3}, -\frac{2}{3}\)

14. \(\frac{5}{4}, -\frac{1}{2}\)

15. \(\frac{3}{7}, \frac{1}{5}\)

16. \(-\frac{7}{8}, \frac{7}{2}\)

17. \(\frac{1}{2}, \frac{3}{4}\)

18. \(\frac{1}{8}, \frac{1}{6}\)
5-3 Study Guide (continued)

Solving Quadratic Equations by Factoring

Solve Equations by Factoring

When you use factoring to solve a quadratic equation, you use the following property.

**Zero Product Property**

For any real numbers \(a\) and \(b\), if \(ab = 0\), then either \(a = 0\) or \(b = 0\), or both \(a\) and \(b = 0\).

**Example**

Solve each equation by factoring.

a. \(3x^2 = 15x\)

\[
3x^2 = 15x \\
3x^2 - 15x = 0 \\
3x(x - 5) = 0 \\
x = 0 \text{ or } x = 5
\]

**Zero Product Property**

The solution set is \(\{0, 5\}\).

b. \(4x^2 - 5x = 21\)

\[
4x^2 - 5x - 21 = 0 \\
(4x + 7)(x - 3) = 0 \\
4x + 7 = 0 \text{ or } x - 3 = 0 \\
x = -\frac{7}{4} \text{ or } x = 3
\]

The solution set is \(\left\{-\frac{7}{4}, 3\right\}\).

**Exercises**

Solve each equation by factoring.

1. \(6x^2 - 2x = 0\)
2. \(x^2 = 7x\)
3. \(20x^2 = -25x\)
4. \(6x^2 = 7x\)
5. \(6x^2 - 27x = 0\)
6. \(12x^2 - 8x = 0\)
7. \(x^2 + x - 30 = 0\)
8. \(2x^2 - x - 3 = 0\)
9. \(x^2 + 14x + 33 = 0\)
10. \(4x^2 + 27x - 7 = 0\)
11. \(3x^2 + 29x - 10 = 0\)
12. \(6x^2 - 5x - 4 = 0\)
13. \(12x^2 - 8x + 1 = 0\)
14. \(5x^2 + 28x - 12 = 0\)
15. \(2x^2 - 250x + 5000 = 0\)
16. \(2x^2 - 11x - 40 = 0\)
17. \(2x^2 + 21x - 11 = 0\)
18. \(3x^2 + 2x - 21 = 0\)
19. \(8x^2 - 14x + 3 = 0\)
20. \(6x^2 + 11x - 2 = 0\)
21. \(5x^2 + 17x - 12 = 0\)
22. \(12x^2 + 25x + 12 = 0\)
23. \(12x^2 + 18x + 6 = 0\)
24. \(7x^2 - 36x + 5 = 0\)
5-3 Practice

Solving Quadratic Equations by Factoring

Write a quadratic equation in standard form with the given root(s).

1. 7, 2
2. 0, 3
3. −5, 8

4. −7, −8
5. −6, −3
6. 3, −4

7. 1, \(\frac{1}{2}\)
8. \(\frac{1}{3}\), 2
9. 0, −\(\frac{7}{2}\)

Factor each polynomial.

10. \(r^3 + 3r^2 - 54r\)
11. \(8a^2 + 2a - 6\)
12. \(c^2 - 49\)

13. \(x^4 + 8\)
14. \(16r^2 - 169\)
15. \(b^4 - 81\)

Solve each equation by factoring.

16. \(x^2 - 4x - 12 = 0\)
17. \(x^2 - 16x + 64 = 0\)
18. \(x^2 - 6x + 8 = 0\)
19. \(x^2 + 3x + 2 = 0\)
20. \(x^2 - 4x = 0\)
21. \(7x^2 = 4x\)
22. \(10x^2 = 9x\)
23. \(x^2 = 2x + 99\)
24. \(x^2 + 12x = -36\)
25. \(5x^2 - 35x + 60 = 0\)
26. \(36x^2 = 25\)
27. \(2x^2 - 8x - 90 = 0\)

28. NUMBER THEORY Find two consecutive even positive integers whose product is 624.

29. NUMBER THEORY Find two consecutive odd positive integers whose product is 323.

30. GEOMETRY The length of a rectangle is 2 feet more than its width. Find the dimensions of the rectangle if its area is 63 square feet.

31. PHOTOGRAPHY The length and width of a 6-inch by 8-inch photograph are reduced by the same amount to make a new photograph whose area is half that of the original. By how many inches will the dimensions of the photograph have to be reduced?
1. **FLASHLIGHTS** When Dora shines her flashlight on the wall at a certain angle, the edge of the lit area is in the shape of a parabola. The equation of the parabola is \( y = 2x^2 + 2x - 60 \). Factor this quadratic equation.

2. **SIGNS** David was looking through an old algebra book and came across this equation.

\[ x^2 + 6x + 8 = 0 \]

The sign in front of the 6 was blotted out. How does the missing sign depend on the signs of the roots?

3. **ART** The area in square inches of the drawing *Maisons prés de la mer* by Claude Monet is approximated by the equation \( y = x^2 - 23x + 130 \). Factor the equation to find the two roots, which are equal to the approximate length and width of the drawing.

4. **PROGRAMMING** Ray is a computer programmer. He needs to find the quadratic function of this graph for an algorithm related to a game involving dice. Provide such a function.

![Graph](image)

5. **ANIMATION** A computer graphics animator would like to make a realistic simulation of a tossed ball. The animator wants the ball to follow the parabolic trajectory represented by the quadratic equation \( f(x) = -0.2(x + 5)(x - 5) \).

a. What are the solutions of \( f(x) = 0 \)?

b. Write \( f(x) \) in standard form.

c. If the animator changes the equation to \( f(x) = -0.2x^2 + 20 \), what are the solutions of \( f(x) = 0 \)?
Completing the Square

**Square Root Property** Use the Square Root Property to solve a quadratic equation that is in the form “perfect square trinomial = constant.”

**Example** Solve each equation by using the Square Root Property. Round to the nearest hundredth if necessary.

a. \(x^2 - 8x + 16 = 25\)
   \[x^2 - 8x + 16 = 25\]
   \[(x - 4)^2 = 25\]
   \[x - 4 = \sqrt{25}\] or \[x - 4 = -\sqrt{25}\]
   \[x = 5 + 4 = 9\] or \[x = -5 + 4 = -1\]
   The solution set is \(\{9, -1\}\).

b. \(4x^2 - 20x + 25 = 32\)
   \[4x^2 - 20x + 25 = 32\]
   \[(2x - 5)^2 = 32\]
   \[2x - 5 = \sqrt{32}\] or \[2x - 5 = -\sqrt{32}\]
   \[x = \frac{5 + 4\sqrt{2}}{2}\]
   The solution set is \(\left\{\frac{5 + 4\sqrt{2}}{2}\right\}\).

**Exercises**

Solve each equation by using the Square Root Property. Round to the nearest hundredth if necessary.

1. \(x^2 - 18x + 81 = 49\)
2. \(x^2 + 20x + 100 = 64\)
3. \(4x^2 + 4x + 1 = 16\)
4. \(36x^2 + 12x + 1 = 18\)
5. \(9x^2 - 12x + 4 = 4\)
6. \(25x^2 + 40x + 16 = 28\)
7. \(4x^2 - 28x + 49 = 64\)
8. \(16x^2 + 24x + 9 = 81\)
9. \(100x^2 - 60x + 9 = 121\)
10. \(25x^2 + 20x + 4 = 75\)
11. \(36x^2 + 48x + 16 = 12\)
12. \(25x^2 - 30x + 9 = 96\)
Completing the Square

**Complete the Square**
To complete the square for a quadratic expression of the form $x^2 + bx$, follow these steps.

1. Find $\frac{b}{2}$.
2. Square $\frac{b}{2}$.
3. Add $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$.

**Example 1**
Find the value of $c$ that makes $x^2 + 22x + c$ a perfect square trinomial. Then write the trinomial as the square of a binomial.

**Step 1**
$b = 22$; $\frac{b}{2} = 11$

**Step 2**
$11^2 = 121$

**Step 3**
$c = 121$

The trinomial is $x^2 + 22x + 121$, which can be written as $(x + 11)^2$.

**Example 2**
Solve $2x^2 - 8x - 24 = 0$ by completing the square.

- Original equation $2x^2 - 8x - 24 = 0$
- Divide each side by 2.
- $\frac{2x^2 - 8x - 24}{2} = 0$
- $x^2 - 4x - 12 = 0$ (not a perfect square)
- $x^2 - 4x = 12$
- Add 12 to each side.
- $(x - 2)^2 = 16$
- Factor the square.
- $x - 2 = \pm 4$
- Square Root Property
- $x = 6$ or $x = -2$
- Solve each equation.

The solution set is $\{6, -2\}$.

**Exercises**

Find the value of $c$ that makes each trinomial a perfect square. Then write the trinomial as a perfect square.

1. $x^2 - 10x + c$
2. $x^2 + 60x + c$
3. $x^2 - 3x + c$

4. $x^2 + 3.2x + c$
5. $x^2 + \frac{1}{2}x + c$
6. $x^2 - 2.5x + c$

Solve each equation by completing the square.

7. $y^2 - 4y - 5 = 0$
8. $x^2 - 8x - 65 = 0$
9. $w^2 - 10w + 21 = 0$

10. $2x^2 - 3x + 1 = 0$
11. $2x^2 - 13x - 7 = 0$
12. $25x^2 + 40x - 9 = 0$

13. $x^2 + 4x + 1 = 0$
14. $y^2 + 12y + 4 = 0$
15. $t^2 + 3t - 8 = 0$
Solve each equation by using the Square Root Property. Round to the nearest hundredth if necessary.

1. \( x^2 + 8x + 16 = 1 \)  
2. \( x^2 + 6x + 9 = 1 \)  
3. \( x^2 + 10x + 25 = 16 \)

4. \( x^2 - 14x + 49 = 9 \)  
5. \( 4x^2 + 12x + 9 = 4 \)  
6. \( x^2 - 8x + 16 = 8 \)

7. \( x^2 - 6x + 9 = 5 \)  
8. \( x^2 - 2x + 1 = 2 \)  
9. \( 9x^2 - 6x + 1 = 2 \)

Find the value of \( c \) that makes each trinomial a perfect square. Then write the trinomial as a perfect square.

10. \( x^2 + 12x + c \)  
11. \( x^2 - 20x + c \)  
12. \( x^2 + 11x + c \)

13. \( x^2 + 0.8x + c \)  
14. \( x^2 - 2.2x + c \)  
15. \( x^2 - 0.36x + c \)

16. \( x^2 + \frac{5}{6}x + c \)  
17. \( x^2 - \frac{1}{4}x + c \)  
18. \( x^2 - \frac{5}{3}x + c \)

Solve each equation by completing the square.

19. \( x^2 + 6x + 8 = 0 \)  
20. \( 3x^2 + x - 2 = 0 \)  
21. \( 3x^2 - 5x + 2 = 0 \)

22. \( x^2 + 18 = 9x \)  
23. \( x^2 - 14x + 19 = 0 \)  
24. \( x^2 + 16x - 7 = 0 \)

25. \( 2x^2 + 8x - 3 = 0 \)  
26. \( x^2 + x - 5 = 0 \)  
27. \( 2x^2 - 10x + 5 = 0 \)

28. \( x^2 + 3x + 6 = 0 \)  
29. \( 2x^2 + 5x + 6 = 0 \)  
30. \( 7x^2 + 6x + 2 = 0 \)

31. **GEOMETRY** When the dimensions of a cube are reduced by 4 inches on each side, the surface area of the new cube is 864 square inches. What were the dimensions of the original cube?

32. **INVESTMENTS** The amount of money \( A \) in an account in which \( P \) dollars are invested for 2 years is given by the formula \( A = P(1 + r)^2 \), where \( r \) is the interest rate compounded annually. If an investment of $800 in the account grows to $882 in two years, at what interest rate was it invested?
5-5 Word Problem Practice

Completing the Square

1. COMPLETING THE SQUARE
Samantha needs to solve the equation
\[ x^2 - 12x = 40. \]
What must she do to each side of the equation to complete the square?

2. ART The area in square inches of the drawing Foliage by Paul Cézanne is approximated by the equation
\[ y = x^2 - 40x + 396. \]
Complete the square and find the two roots, which are equal to the approximate length and width of the drawing.

3. COMPOUND INTEREST Nikki invested $1000 in a savings account with interest compounded annually. After two years the balance in the account is $1210. Use the compound interest formula
\[ A = P(1 + r)^t \]
to find the annual interest rate.

4. REACTION TIME Lauren was eating lunch when she saw her friend Jason approach. The room was crowded and Jason had to lift his tray to avoid obstacles. Suddenly, a glass on Jason’s lunch tray tipped and fell off the tray. Lauren lunged forward and managed to catch the glass just before it hit the ground. The height \( h \), in feet, of the glass \( t \) seconds after it was dropped is given by \( h = -16t^2 + 4.5 \). Lauren caught the glass when it was six inches off the ground. How long was the glass in the air before Lauren caught it?

5. PARABOLAS A parabola is modeled by
\[ y = x^2 - 10x + 28. \]
Jane’s homework problem requires that she find the vertex of the parabola. She uses the completing square method to express the function in the form
\[ y = (x - h)^2 + k, \]
where \((h, k)\) is the vertex of the parabola. Write the function in the form used by Jane.

6. AUDITORIUM SEATING The seats in an auditorium are arranged in a square grid pattern. There are 45 rows and 45 columns of chairs. For a special concert, organizers decide to increase seating by adding \( n \) rows and \( n \) columns to make a square pattern of seating \( 45 + n \) seats on a side.

a. How many seats are there after the expansion?

b. What is \( n \) if organizers wish to add 1000 seats?

c. If organizers do add 1000 seats, what is the seating capacity of the auditorium?
The Quadratic Formula and the Discriminant

**Quadratic Formula**  
The Quadratic Formula can be used to solve any quadratic equation once it is written in the form $ax^2 + bx + c = 0$.  

<table>
<thead>
<tr>
<th>Quadratic Formula</th>
<th>The solutions of $ax^2 + bx + c = 0$, with $a \neq 0$, are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.</th>
</tr>
</thead>
</table>

**Example**  
Solve $x^2 - 5x = 14$ by using the Quadratic Formula.

Rewrite the equation as $x^2 - 5x - 14 = 0$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$  

Quadratic Formula

$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-14)}}{2(1)}$  

Replace $a$ with 1, $b$ with $-5$, and $c$ with $-14$.

$= \frac{5 \pm \sqrt{81}}{2}$  

Simplify.

$= \frac{5 \pm 9}{2}$

$= \frac{5 + 9}{2}$  

$= \frac{5 - 9}{2}$

$= 7$ or $-2$

The solutions are $-2$ and $7$.

**Exercises**

Solve each equation by using the Quadratic Formula.

1. $x^2 + 2x - 35 = 0$
2. $x^2 + 10x + 24 = 0$
3. $x^2 - 11x + 24 = 0$

4. $4x^2 + 19x - 5 = 0$
5. $14x^2 + 9x + 1 = 0$
6. $2x^2 - x - 15 = 0$

7. $3x^2 + 5x = 2$
8. $2y^2 + y - 15 = 0$
9. $3x^2 - 16x + 16 = 0$

10. $8x^2 + 6x - 9 = 0$
11. $r^2 - \frac{3r}{5} + \frac{2}{25} = 0$
12. $x^2 - 10x - 50 = 0$

13. $x^2 + 6x - 23 = 0$
14. $4x^2 - 12x - 63 = 0$
15. $x^2 - 6x + 21 = 0$
Roots and the Discriminant

The expression under the radical sign, \(b^2 - 4ac\), in the Quadratic Formula is called the **discriminant**.

<table>
<thead>
<tr>
<th>Discriminant</th>
<th>Type and Number of Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b^2 - 4ac &gt; 0) and a perfect square</td>
<td>2 rational roots</td>
</tr>
<tr>
<td>(b^2 - 4ac &gt; 0), but not a perfect square</td>
<td>2 irrational roots</td>
</tr>
<tr>
<td>(b^2 - 4ac = 0)</td>
<td>1 rational root</td>
</tr>
<tr>
<td>(b^2 - 4ac &lt; 0)</td>
<td>2 complex roots</td>
</tr>
</tbody>
</table>

**Example**

Find the value of the discriminant for each equation. Then describe the number and type of roots for the equation.

- **a.** \(2x^2 + 5x + 3\)
  
  The discriminant is \(b^2 - 4ac = 5^2 - 4(2)(3)\) or 1.
  
  The discriminant is a perfect square, so the equation has 2 rational roots.

- **b.** \(3x^2 - 2x + 5\)
  
  The discriminant is \(b^2 - 4ac = (-2)^2 - 4(3)(5)\) or -56.
  
  The discriminant is negative, so the equation has 2 complex roots.

**Exercises**

Complete parts a–c for each quadratic equation.

- a. Find the value of the discriminant.
- b. Describe the number and type of roots.
- c. Find the exact solutions by using the Quadratic Formula.

1. \(p^2 + 12p = -4\)

2. \(9x^2 - 6x + 1 = 0\)

3. \(2x^2 - 7x - 4 = 0\)

4. \(x^2 + 4x - 4 = 0\)

5. \(5x^2 - 36x + 7 = 0\)

6. \(4x^2 - 4x + 11 = 0\)

7. \(x^2 - 7x + 6 = 0\)

8. \(m^2 - 8m = -14\)

9. \(25x^2 - 40x = -16\)

10. \(4x^2 + 20x + 29 = 0\)

11. \(6x^2 + 26x + 8 = 0\)

12. \(4x^2 - 4x - 11 = 0\)
5-6 Practice

The Quadratic Formula and the Discriminant

Solve each equation by using the Quadratic Formula.

1. \(7x^2 - 5x = 0\)
2. \(4x^2 - 9 = 0\)
3. \(3x^2 + 8x = 3\)
4. \(x^2 - 21 = 4x\)
5. \(3x^2 - 13x + 4 = 0\)
6. \(15x^2 + 22x = -8\)
7. \(x^2 - 6x + 3 = 0\)
8. \(x^2 - 14x + 53 = 0\)
9. \(3x^2 = -54\)
10. \(25x^2 - 20x - 6 = 0\)
11. \(4x^2 - 4x + 17 = 0\)
12. \(8x - 1 = 4x^2\)
13. \(x^2 = 4x - 15\)
14. \(4x^2 - 12x + 7 = 0\)

Complete parts a–c for each quadratic equation.

a. Find the value of the discriminant.
b. Describe the number and type of roots.
c. Find the exact solutions by using the Quadratic Formula.

15. \(x^2 - 16x + 64 = 0\)
16. \(x^2 = 3x\)
17. \(9x^2 - 24x + 16 = 0\)
18. \(x^2 - 3x = 40\)
19. \(3x^2 + 9x - 2 = 0\)
20. \(2x^2 + 7x = 0\)
21. \(5x^2 - 2x + 4 = 0\)
22. \(12x^2 - x - 6 = 0\)
23. \(7x^2 + 6x + 2 = 0\)
24. \(12x^2 + 2x - 4 = 0\)
25. \(6x^2 - 2x - 1 = 0\)
26. \(x^2 + 3x + 6 = 0\)
27. \(4x^2 - 3x^2 - 6 = 0\)
28. \(16x^2 - 8x + 1 = 0\)
29. \(2x^2 - 5x - 6 = 0\)

30. GRAVITATION The height \(h(t)\) in feet of an object \(t\) seconds after it is propelled straight up from the ground with an initial velocity of 60 feet per second is modeled by the equation \(h(t) = -16t^2 + 60t\). At what times will the object be at a height of 56 feet?

31. STOPPING DISTANCE The formula \(d = 0.05s^2 + 1.1s\) estimates the minimum stopping distance \(d\) in feet for a car traveling \(s\) miles per hour. If a car stops in 200 feet, what is the fastest it could have been traveling when the driver applied the brakes?
1. **PARABOLAS** The graph of a quadratic equation of the form \( y = ax^2 + bx + c \) is shown below.

![Graph of parabola](image)

Is the discriminant \( b^2 - 4ac \) positive, negative, or zero? Explain.

2. **TANGENT** Kathleen is trying to find \( b \) so that the \( x \)-axis is tangent to the parabola \( y = x^2 + bx + 4 \). She finds one value that works, \( b = 4 \). Is this the only value that works? Explain.

3. **SPORTS** In 1990, American Randy Barnes set the world record for the shot put. His throw can be described by the equation \( y = -16x^2 + 368x \). Use the Quadratic Formula to find how far his throw was to the nearest foot.

4. **EXAMPLES** Give an example of a quadratic function \( f(x) \) that has the following properties.
   
   I. The discriminant of \( f \) is zero.
   
   II. There is no real solution of the equation \( f(x) = 10 \).

   Sketch the graph of \( x = f(x) \).

5. **TANGENTS** The graph of \( y = x^2 \) is a parabola that passes through the point at \((1, 1)\). The line \( y = mx - m + 1 \), where \( m \) is a constant, also passes through the point at \((1, 1)\).

   a. To find the points of intersection between the line \( y = mx - m + 1 \) and the parabola \( y = x^2 \), set \( x^2 = mx - m + 1 \) and then solve for \( x \). Rearranging terms, this equation becomes \( x^2 - mx + m - 1 = 0 \). What is the discriminant of this equation?

   b. For what value of \( m \) is there only one point of intersection? Explain the meaning of this in terms of the corresponding line and the parabola.
Transformations with Quadratic Functions

Write Quadratic Equations in Vertex Form A quadratic function is easier to graph when it is in vertex form. You can write a quadratic function of the form \( y = ax^2 + bx + c \) in vertex form by completing the square.

**Example** Write \( y = 2x^2 - 12x + 25 \) in vertex form. Then graph the function.

\[
y = 2x^2 - 12x + 25 \\
y = 2(x^2 - 6x) + 25 \\
y = 2(x^2 - 6x + 9) + 25 - 18 \\
y = 2(x - 3)^2 + 7
\]

The vertex form of the equation is \( y = 2(x - 3)^2 + 7 \).

**Exercises**

Write each equation in vertex form. Then graph the function.

1. \( y = x^2 - 10x + 32 \)
2. \( y = x^2 + 6x \)
3. \( y = x^2 - 8x + 6 \)
4. \( y = -4x^2 + 16x - 11 \)
5. \( y = 3x^2 - 12x + 5 \)
6. \( y = 5x^2 - 10x + 9 \)
Transformations of Quadratic Functions

Parabolas can be transformed by changing the values of the constants \(a\), \(h\), and \(k\) in the vertex form of a quadratic equation:

\[ y = a(x - h)^2 + k. \]

- The sign of \(a\) determines whether the graph opens upward \((a > 0)\) or downward \((a < 0)\).
- The absolute value of \(a\) also causes a dilation (enlargement or reduction) of the parabola. The parabola becomes narrower if \(|a| > 1\) and wider if \(|a| < 1\).
- The value of \(h\) translates the parabola horizontally. Positive values of \(h\) slide the graph to the right and negative values slide the graph to the left.
- The value of \(k\) translates the graph vertically. Positive values of \(k\) slide the graph upward and negative values slide the graph downward.

Example

Graph \(y = (x + 7)^2 + 3\).

- Rewrite the equation as \(y = [x - (-7)]^2 + 3\).
- Because \(h = -7\) and \(k = 3\), the vertex is at \((-7, 3)\). The axis of symmetry is \(x = -7\). Because \(a = 1\), we know that the graph opens up, and the graph is the same width as the graph of \(y = x^2\).
- Translate the graph of \(y = x^2\) seven units to the left and three units up.

Exercises

Graph each function.

1. \(y = -2x^2 + 2\)
2. \(y = -3(x - 1)^2\)
3. \(y = 2(x + 2)^2 + 3\)
5-7 Practice

Transformations with Quadratic Functions

Write each equation in vertex form. Then identify the vertex, axis of symmetry, and direction of opening.

1. \( y = -6x^2 - 24x - 25 \)
2. \( y = 2x^2 + 2 \)
3. \( y = -4x^2 + 8x \)

4. \( y = x^2 + 10x + 20 \)
5. \( y = 2x^2 + 12x + 18 \)
6. \( y = 3x^2 - 6x + 5 \)

7. \( y = -2x^2 - 16x - 32 \)
8. \( y = -3x^2 + 18x - 21 \)
9. \( y = 2x^2 + 16x + 29 \)

Graph each function.

10. \( y = (x + 3)^2 - 1 \)
11. \( y = -x^2 + 6x - 5 \)
12. \( y = 2x^2 - 2x + 1 \)

13. Write an equation for a parabola with vertex at \((1, 3)\) that passes through \((-2, -15)\).

14. Write an equation for a parabola with vertex at \((-3, 0)\) that passes through \((3, 18)\).

15. BASEBALL The height \( h \) of a baseball \( t \) seconds after being hit is given by \( h(t) = -16t^2 + 80t + 3 \). What is the maximum height that the baseball reaches, and when does this occur?

16. SCULPTURE A modern sculpture in a park contains a parabolic arc that starts at the ground and reaches a maximum height of 10 feet after a horizontal distance of 4 feet. Write a quadratic function in vertex form that describes the shape of the outside of the arc, where \( y \) is the height of a point on the arc and \( x \) is its horizontal distance from the left-hand starting point of the arc.
5-7 Word Problem Practice

Transformations with Quadratic Functions

1. ARCHES A parabolic arch is used as a bridge support. The graph of the arch is shown below.

If the equation that corresponds to this graph is written in the form \( y + a(x - h)^2 + k \), what are \( h \) and \( k \)?

2. TRANSLATIONS For a computer animation, Barbara uses the quadratic function \( f(x) = -42(x - 20)^2 + 16800 \) to help her simulate an object tossed on another planet. For one skit, she had to use the function \( f(x + 5) - 8000 \) instead of \( f(x) \). Where is the vertex of the graph of \( y = f(x + 5) - 8000 \)?

3. BRIDGES The shape formed by the main cables of the Golden Gate Bridge approximately follows the equation \( y = 0.0002x^2 - 0.23x + 227 \). Graph the parabola formed by one of the cables.

4. WATER JETS The graph shows the path of a jet of water.

The equation corresponding to this graph is \( y = a(x - h)^2 + k \). What are \( a \), \( h \), and \( k \)?

5. PROFIT A theater operator predicts that the theater can make \(-4x^2 + 160x\) dollars per show if tickets are priced at \( x \) dollars.

   a. Rewrite the equation \( y = -4x^2 + 160x \) in the form \( y = a(x - h)^2 + k \).

   b. What is the vertex of the parabola and what is its axis of symmetry?

   c. Graph the parabola.
Polynomial Functions

A polynomial of degree \( n \) in one variable \( x \) is an expression of the form
\[
a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,
\]
where the coefficients \( a_{n-1}, a_{n-2}, a_{n-3}, \ldots, a_0 \) represent real numbers, \( a_n \) is not zero, and \( n \) represents a nonnegative integer.

The degree of a polynomial in one variable is the greatest exponent of its variable. The leading coefficient is the coefficient of the term with the highest degree.

Example 1
What are the degree and leading coefficient of \( 3x^2 - 2x^4 - 7 + x^3 \)?

Rewrite the expression so the powers of \( x \) are in decreasing order.
\[-2x^4 + x^3 + 3x^2 - 7\]

This is a polynomial in one variable. The degree is 4, and the leading coefficient is \(-2\).

Example 2
Find \( f(-5) \) if \( f(x) = x^3 + 2x^2 - 10x + 20 \).
\[
\begin{align*}
f(x) &= x^3 + 2x^2 - 10x + 20 \\
f(-5) &= (-5)^3 + 2(-5)^2 - 10(-5) + 20 \\
&= -125 + 50 + 50 + 20 \\
&= -5
\end{align*}
\]

Evaluate.

Example 3
Find \( g(a^2 - 1) \) if \( g(x) = x^2 + 3x - 4 \).
\[
\begin{align*}
g(x) &= x^2 + 3x - 4 \\
g(a^2 - 1) &= (a^2 - 1)^2 + 3(a^2 - 1) - 4 \\
&= a^4 - 2a^2 + 1 + 3a^2 - 3 - 4 \\
&= a^4 + a^2 - 6
\end{align*}
\]

Simplify.

Exercises
State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

1. \( 3x^4 + 6x^3 - x^2 + 12 \)
2. \( 100 - 5x^3 + 10x^7 \)
3. \( 4x^6 + 6x^4 + 8x^8 - 10x^2 + 20 \)
4. \( 4x^2 - 3xy + 16y^2 \)
5. \( 8x^3 - 9x^5 + 4x^2 - 36 \)
6. \( \frac{x^2}{18} - \frac{x^6}{25} + \frac{x^3}{36} - \frac{1}{72} \)

Find \( f(2) \) and \( f(-5) \) for each function.

7. \( f(x) = x^2 - 9 \)
8. \( f(x) = 4x^3 - 3x^2 + 2x - 1 \)
9. \( f(x) = 9x^3 - 4x^2 + 5x + 7 \)
Graphs of Polynomial Functions

### End Behavior of Polynomial Functions

- If the degree is even and the leading coefficient is positive, then
  \[ f(x) \to +\infty \text{ as } x \to -\infty \quad f(x) \to +\infty \text{ as } x \to +\infty \]
- If the degree is even and the leading coefficient is negative, then
  \[ f(x) \to -\infty \text{ as } x \to -\infty \quad f(x) \to -\infty \text{ as } x \to +\infty \]
- If the degree is odd and the leading coefficient is positive, then
  \[ f(x) \to -\infty \text{ as } x \to -\infty \quad f(x) \to +\infty \text{ as } x \to +\infty \]
- If the degree is odd and the leading coefficient is negative, then
  \[ f(x) \to +\infty \text{ as } x \to -\infty \quad f(x) \to -\infty \text{ as } x \to +\infty \]

### Real Zeros of a Polynomial Function

The maximum number of zeros of a polynomial function is equal to the degree of the polynomial. A zero of a function is a point at which the graph intersects the \(x\)-axis. On a graph, count the number of real zeros of the function by counting the number of times the graph crosses or touches the \(x\)-axis.

### Example

Determine whether the graph represents an odd-degree polynomial or an even-degree polynomial. Then state the number of real zeros.

As \(x \to -\infty\), \(f(x) \to -\infty\) and as \(x \to +\infty\), \(f(x) \to +\infty\), so it is an odd-degree polynomial function. The graph intersects the \(x\)-axis at 1 point, so the function has 1 real zero.

### Exercises

For each graph,

- **a.** describe the end behavior,
- **b.** determine whether it represents an odd-degree or an even-degree function, and
- **c.** state the number of real zeroes.

1. 

2. 

3. 

6-3 Practice

Polynomial Functions

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

1. \((3x^2 + 1)(2x^2 - 9)\)

2. \(\frac{1}{5}a^3 - \frac{3}{5}a^2 + \frac{4}{5}a\)

3. \(\frac{2}{m^2} + 3m - 12\)

4. \(27 + 3xy^3 - 12x^2y^2 - 10y\)

Find \(p(-2)\) and \(p(3)\) for each function.

5. \(p(x) = x^3 - x^5\)

6. \(p(x) = -7x^2 + 5x + 9\)

7. \(p(x) = -x^5 + 4x^3\)

8. \(p(x) = 3x^3 - x^2 + 2x - 5\)

9. \(p(x) = x^4 + \frac{1}{2}x^3 - \frac{1}{2}x\)

10. \(p(x) = \frac{1}{3x^3} + \frac{2}{3x^2} + 3x\)

If \(p(x) = 3x^2 - 4\) and \(r(x) = 2x^2 - 5x + 1\), find each value.

11. \(p(8a)\)

12. \(r(a^2)\)

13. \(-5r(2a)\)

14. \(r(x + 2)\)

15. \(p(x^2 - 1)\)

16. \(5p(x + 2)\)

For each graph,

a. describe the end behavior,

b. determine whether it represents an odd-degree or an even-degree function, and

c. state the number of real zeroes.

17. 

18. 

19. 

20. WIND CHILL The function \(C(w) = 0.013w^2 - w - 7\) estimates the wind chill temperature \(C(w)\) at 0°F for wind speeds \(w\) from 5 to 30 miles per hour. Estimate the wind chill temperature at 0°F if the wind speed is 20 miles per hour.
1. **MANUFACTURING** A metal sheet is curved according to the shape of the graph of \( f(x) = x^4 - 9x^2 \). What is the degree of this polynomial?

2. **GRAPHS** Kendra graphed the polynomial \( f(x) \) shown below.

3. **PENTAGONAL NUMBERS** The \( n \)th pentagonal number is given by the expression

\[
\frac{n(3n - 1)}{2}.
\]

What is the degree of this polynomial? What is the seventh pentagonal number?

4. **DRILLING** The volume of a drill bit can be estimated by the formula for a cone,

\[
V = \frac{1}{3} \pi h r^2,
\]

where \( h \) is the height of the bit and \( r \) is its radius. Substituting \( \frac{\sqrt{3}}{3} r \) for \( h \), the volume of the drill bit is estimated as \( \frac{\sqrt{3}}{9} \pi r^3 \). Graph the function of drill bit volume. Describe the end behavior, degree, and sign of the leading coefficient.

5. **TRIANGLES** Dylan drew \( n \) dots on a piece of paper making sure that no line contained 3 of the dots. The number of triangles that can be made using the dots as vertices is equal to

\[
f(n) = \frac{1}{6}(n^3 - 3n^2 + 2n).
\]

a. What is the degree of \( f \)?

b. If Dylan drew 15 dots, how many triangles can be made?
Graphs of Polynomial Functions

Location Principle

Suppose \( y = f(x) \) represents a polynomial function and \( a \) and \( b \) are two numbers such that \( f(a) < 0 \) and \( f(b) > 0 \). Then the function has at least one real zero between \( a \) and \( b \).

Example

Determine consecutive integer values of \( x \) between which each real zero of \( f(x) = 2x^4 - x^3 - 5 \) is located. Then draw the graph.

Make a table of values. Look at the values of \( f(x) \) to locate the zeros. Then use the points to sketch a graph of the function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>35</td>
</tr>
<tr>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td>-5</td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>19</td>
</tr>
</tbody>
</table>

The changes in sign indicate that there are zeros between \( x = -2 \) and \( x = -1 \) and between \( x = 1 \) and \( x = 2 \).

Exercises

Graph each function by making a table of values. Determine the values of \( x \) between which each real zero is located.

1. \( f(x) = x^3 - 2x^2 + 1 \)
2. \( f(x) = x^4 + 2x^3 - 5 \)
3. \( f(x) = -x^4 + 2x^2 - 1 \)
4. \( f(x) = x^3 - 3x^2 + 4 \)
5. \( f(x) = 3x^3 + 2x - 1 \)
6. \( f(x) = x^4 - 3x^3 + 1 \)
Maximum and Minimum Points

A quadratic function has either a maximum or a minimum point on its graph. For higher degree polynomial functions, you can find turning points, which represent relative maximum or relative minimum points.

**Example**

Graph \( f(x) = x^3 + 6x^2 - 3 \). Estimate the \( x \)-coordinates at which the relative maxima and minima occur.

Make a table of values and graph the function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>22</td>
</tr>
<tr>
<td>-4</td>
<td>29</td>
</tr>
<tr>
<td>-3</td>
<td>24</td>
</tr>
<tr>
<td>-2</td>
<td>13</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>-3</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>29</td>
</tr>
</tbody>
</table>

A relative maximum occurs at \( x = -4 \) and a relative minimum occurs at \( x = 0 \).

### Exercises

Graph each polynomial function. Estimate the \( x \)-coordinates at which the relative maxima and relative minima occur.

1. \( f(x) = x^3 - 3x^2 \)
2. \( f(x) = 2x^3 + x^2 - 3x \)
3. \( f(x) = 2x^3 - 3x + 2 \)
4. \( f(x) = x^4 - 7x - 3 \)
5. \( f(x) = x^5 - 2x^2 + 2 \)
6. \( f(x) = x^3 + 2x^2 - 3 \)
Complete each of the following.
a. Graph each function by making a table of values.
b. Determine the consecutive values of \( x \) between which each real zero is located.
c. Estimate the \( x \)-coordinates at which the relative maxima and minima occur.

1. \( f(x) = -x^3 + 3x^2 - 3 \)

\[
\begin{array}{c|c}
 x & f(x) \\
-2 & \ \\
-1 & \ \\
0 & \ \\
1 & \ \\
2 & \ \\
3 & \ \\
4 & \ \\
\end{array}
\]

2. \( f(x) = x^3 - 1.5x^2 - 6x + 1 \)

\[
\begin{array}{c|c}
 x & f(x) \\
-2 & \ \\
-1 & \ \\
0 & \ \\
1 & \ \\
2 & \ \\
3 & \ \\
4 & \ \\
\end{array}
\]

3. \( f(x) = 0.75x^4 + x^3 - 3x^2 + 4 \)

\[
\begin{array}{c|c}
 x & f(x) \\
-2 & \ \\
-1 & \ \\
0 & \ \\
1 & \ \\
2 & \ \\
3 & \ \\
4 & \ \\
\end{array}
\]

4. \( f(x) = x^4 + 4x^3 + 6x^2 + 4x - 3 \)

\[
\begin{array}{c|c}
 x & f(x) \\
-2 & \ \\
-1 & \ \\
0 & \ \\
1 & \ \\
2 & \ \\
3 & \ \\
4 & \ \\
\end{array}
\]

5. **PRICES** The Consumer Price Index (CPI) gives the relative price for a fixed set of goods and services. The CPI from September, 2000 to July, 2001 is shown in the graph.

   **Source:** U. S. Bureau of Labor Statistics

   a. Describe the turning points of the graph.

   b. If the graph were modeled by a polynomial equation, what is the least degree the equation could have?

6. **LABOR** A town’s jobless rate can be modeled by \((1, 3.3), (2, 4.9), (3, 5.3), (4, 6.4), (5, 4.5), (6, 5.6), (7, 2.5), \) and \((8, 2.7)\). How many turning points would the graph of a polynomial function through these points have? Describe them.
1. LANDSCAPES Jalen uses a fourth-degree polynomial to describe the shape of two hills in the background of a video game that he is helping to write. The graph of the polynomial is shown below.

Estimate the x-coordinates at which the relative maxima and relative minima occur.

2. NATIONAL PARKS The graph models the cross-section of Mount Rushmore.

What is the smallest degree possible for the equation that corresponds with this graph?

3. VALUE A banker models the expected value of a company in millions of dollars by the formula \( n^3 - 3n^2 \), where \( n \) is the number of years in business. Sketch a graph of \( v = n^3 - 3n^2 \).

4. CONSECUTIVE NUMBERS Ms. Sanchez asks her students to write expressions to represent five consecutive integers. One solution is \( x - 2, x - 1, x, x + 1, \) and \( x + 2 \). The product of these five consecutive integers is given by the fifth degree polynomial \( f(x) = x^5 - 5x^3 + 4x \).

a. For what values of \( x \) is \( f(x) = 0 \)?

b. Sketch the graph of \( y = f(x) \).
Roots and Zeros

Synthetic Types of Roots The following statements are equivalent for any polynomial function \( f(x) \).

- \( c \) is a zero of the polynomial function \( f(x) \).
- \( c \) is a root or solution of the polynomial equation \( f(x) = 0 \).
- \( (x - c) \) is a factor of the polynomial \( f(x) \).
- If \( c \) is real, then \((c, 0)\) is an intercept of the graph of \( f(x) \).

<table>
<thead>
<tr>
<th>Fundamental Theorem of Algebra</th>
<th>Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corollary to the Fundamental Theorem of Algebras</td>
<td>A polynomial equation of the form ( P(x) = 0 ) of degree ( n ) with complex coefficients has exactly ( n ) roots in the set of complex numbers, including repeated roots.</td>
</tr>
</tbody>
</table>
| Descartes’ Rule of Signs | If \( P(x) \) is a polynomial with real coefficients whose terms are arranged in descending powers of the variable,  
- the number of positive real zeros of \( y = P(x) \) is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and  
- the number of negative real zeros of \( y = P(x) \) is the same as the number of changes in sign of the coefficients of the terms of \( P(-x) \), or is less than this number by an even number. |

**Example 1** Solve the equation \( 6x^3 + 3x = 0 \). State the number and type of roots.

\[
6x^3 + 3x = 0 \\
3x(2x^2 + 1) = 0 \\
Use the Zero Product Property.
\]

\[
x = 0 \quad \text{or} \quad 2x^2 + 1 = 0
\]

\[
x = 0 \quad \text{or} \quad 2x^2 = -1
\]

\[
x = \pm \frac{i\sqrt{2}}{2}
\]

The equation has one real root, 0, and two imaginary roots, \( \pm \frac{i\sqrt{2}}{2} \).

**Example 2** State the number of positive real zeros, negative real zeros, and imaginary zeros for \( p(x) = 4x^4 - 3x^3 - x^2 + 2x - 5 \).

Since \( p(x) \) has degree 4, it has 4 zeros.

Since there are three sign changes, there are 3 or 1 positive real zeros.

Find \( p(-x) \) and count the number of changes in sign for its coefficients.

\[
p(-x) = 4(-x)^4 - 3(-x)^3 + (-x)^2 + 2(-x) - 5 \\
= 4x^4 + 3x^3 + x^2 - 2x - 5
\]

Since there is one sign change, there is exactly 1 negative real zero.

Thus, there are 3 positive and 1 negative real zero or 1 positive and 1 negative real zeros and 2 imaginary zeros.

**Exercises**

Solve each equation. State the number and type of roots.

1. \( x^2 + 4x - 21 = 0 \)
2. \( 2x^3 - 50x = 0 \)
3. \( 12x^3 + 100x = 0 \)

State the possible number of positive real zeros, negative real zeros, and imaginary zeros for each function.

4. \( f(x) = 3x^3 + x^2 - 8x - 12 \)
5. \( f(x) = 3x^5 - x^4 - x^3 + 6x^2 - 5 \)
Roots and Zeros

Find Zeros

Complex Conjugate Theorem
Suppose $a$ and $b$ are real numbers with $b \neq 0$. If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.

Example
Find all of the zeros of $f(x) = x^4 - 15x^2 + 38x - 60$.

Since $f(x)$ has degree 4, the function has 4 zeros.

$f(x) = x^4 - 15x^2 + 38x - 60$
$f(-x) = x^4 - 15x^2 - 38x - 60$

Since there are 3 sign changes for the coefficients of $f(x)$, the function has 3 or 1 positive real zeros. Since there is + sign change for the coefficients of $f(-x)$, the function has 1 negative real zero. Use synthetic substitution to test some possible zeros.

$$\begin{array}{c|cccc}
    2 & 1 & 0 & -15 & 38 & -60 \\
    & 2 & 4 & -22 & 32 \\
\hline
    1 & 2 & -11 & 16 & -28 \\
\end{array}$$

$$\begin{array}{c|cccc}
    3 & 1 & 0 & -15 & 38 & -60 \\
    & 3 & 9 & -18 & 60 \\
\hline
    1 & 3 & -6 & 20 & 0 \\
\end{array}$$

So 3 is a zero of the polynomial function. Now try synthetic substitution again to find a zero of the depressed polynomial.

$$\begin{array}{c|cccc}
    -2 & 1 & 3 & -6 & 20 \\
    & -2 & -2 & 16 \\
\hline
    1 & 1 & -8 & 36 \\
\end{array}$$

$$\begin{array}{c|cccc}
    -4 & 1 & 3 & -6 & 20 \\
    & -4 & 4 & 8 \\
\hline
    1 & -1 & -2 & 28 \\
\end{array}$$

$$\begin{array}{c|cccc}
    -5 & 1 & 3 & -6 & 20 \\
    & -5 & 10 & -20 \\
\hline
    1 & -2 & 4 & 0 \\
\end{array}$$

So $-5$ is another zero. Use the Quadratic Formula on the depressed polynomial $x^2 - 2x + 4$ to find the other 2 zeros, $1 \pm i\sqrt{3}$.

The function has two real zeros at 3 and $-5$ and two imaginary zeros at $1 \pm i\sqrt{3}$.

Exercises
Find all zeros of each function.

1. $f(x) = x^3 + x^2 + 9x + 9$
2. $f(x) = x^3 - 3x^2 + 4x - 12$

3. $p(a) = a^3 - 10a^2 + 34a - 40$
4. $p(x) = x^3 - 5x^2 + 11x - 15$

5. $f(x) = x^3 + 6x + 20$
6. $f(x) = x^4 - 3x^3 + 21x^2 - 75x - 100$
Practice

Roots and Zeros

Solve each equation. State the number and type of roots.

1. \(-9x - 15 = 0\)  
2. \(x^4 - 5x^2 + 4 = 0\)

3. \(x^3 - 81x = 0\)  
4. \(x^3 + x^2 - 3x - 3 = 0\)

5. \(x^3 + 6x + 20 = 0\)  
6. \(x^4 - x^3 - x^2 - x - 2 = 0\)

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

7. \(f(x) = 4x^3 - 2x^2 + x + 3\)  
8. \(p(x) = 2x^4 - 2x^3 + 2x^2 - x - 1\)

9. \(q(x) = 3x^4 + x^3 - 3x^2 + 7x + 5\)  
10. \(h(x) = 7x^4 + 3x^3 - 2x^2 - x + 1\)

Find all zeros of each function.

11. \(h(x) = 2x^3 + 3x^2 - 65x + 84\)  
12. \(p(x) = x^3 - 3x^2 + 9x - 7\)

13. \(h(x) = x^3 - 7x^2 + 17x - +5\)  
14. \(q(x) = x^4 + 50x^2 + 49\)

15. \(g(x) = x^4 + 4x^3 - 3x^2 - 14x - 8\)  
16. \(f(x) = x^4 - 6x^3 + 6x^2 + 24x - 40\)

Write a polynomial function of least degree with integral coefficients that has the given zeros.

17. \(-5, 3i\)  
18. \(-2, 3 + i\)

19. \(-1, 4, 3i\)  
20. \(2, 5, 1 + i\)

21. CRAFTS Stephan has a set of plans to build a wooden box. He wants to reduce the volume of the box to 105 cubic inches. He would like to reduce the length of each dimension in the plan by the same amount. The plans call for the box to be 10 inches by 8 inches by 6 inches. Write and solve a polynomial equation to find out how much Stephan should take from each dimension.
1. **TABLES**  Li Pang made a table of values for the polynomial \( p(x) \). Her table is shown below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( p(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>-3</td>
</tr>
<tr>
<td>-3</td>
<td>-1</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Name three roots of \( p(x) \).

2. **ROOTS**  Ryan is an electrical engineer. He often solves polynomial equations to work out various properties of the circuits he builds. For one circuit, he must find the roots of a polynomial \( p(x) \). He finds that \( p(2 - 3i) = 0 \). Give two different roots of \( p(x) \).

3. **REAL ROOTS**  There are more than a thousand roller coasters around the world. Roller coaster designers can use polynomial functions to model the shapes of possible roller coasters. Madison is studying a roller coaster modeled by the polynomial \( f(x) = x^6 - 14x^4 + 49x^2 - 36 \). She knows that all of the roots of \( f(x) \) are real. How many positive and how many negative roots are there? How are the set of positive roots and negative roots related to each other? Explain.

4. **COMPLEX ROOTS**  Eric is a statistician. During the course of his work, he had to find something called the “eigenvalues of a matrix,” which was basically the same as finding the roots of a polynomial. The polynomial was \( x^4 + 6x^2 + 25 \). One of the roots of this polynomial is \( 1 + 2i \). What are the other 3 roots? Explain.

5. **QUADRILATERALS**  Shayna plotted the four vertices of a quadrilateral in the complex plane and then encoded the points in a polynomial \( p(x) \) by making them the roots of \( p(x) \). The polynomial \( p(x) \) is \( x^4 - 9x^3 + 27x^2 + 23x - 150 \).

   a. The polynomial \( p(x) \) has one positive real root, and it is an integer. Find the integer.

   b. Find the negative real root(s) of \( p(x) \).

   c. Find the complex roots of \( p(x) \).
6-8 Study Guide

Rational Zero Theorem

Identify Rational Zeros

| Rational Zero Theorem | Let \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 \) represent a polynomial function with integral coefficients. If \( \frac{p}{q} \) is a rational number in simplest form and is a zero of \( y = f(x) \), then \( p \) is a factor of \( a_n \) and \( q \) is a factor of \( a_0 \).
| Corollary (Integral Zero Theorem) | If the coefficients of a polynomial are integers such that \( a_n = 1 \) and \( a_0 \neq 0 \), any rational zeros of the function must be factors of \( a_0 \).

Example

List all of the possible rational zeros of each function.

a. \( f(x) = 3x^3 - 2x^2 + 6x - 10 \)
   
   If \( \frac{p}{q} \) is a rational root, then \( p \) is a factor of \(-10\) and \( q \) is a factor of \( 3 \). The possible values for \( p \) are \( \pm 1, \pm 2, \pm 5, \) and \( \pm 10 \). The possible values for \( q \) are \( 61 \) and \( 63 \). So all of the possible rational zeros are \( \frac{p}{q} = \pm 1, \pm 2, \pm 5, \pm 10, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{5}{3} \), and \( \pm \frac{10}{3} \).

b. \( g(x) = x^3 - 10x^2 + 14x - 36 \)
   
   Since the coefficient of \( x^3 \) is \( 1 \), the possible rational zeros must be the factors of the constant term \(-36\). So the possible rational zeros are \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \) and \( \pm 36 \).

Exercises

List all of the possible rational zeros of each function.

1. \( f(x) = x^3 + 3x^2 - x + 8 \)
2. \( g(x) = x^5 - 7x^4 + 3x^2 + x - 20 \)
3. \( h(x) = x^4 - 7x^3 - 4x^2 + x - 49 \)
4. \( p(x) = 2x^4 - 5x^3 + 8x^2 + 3x - 5 \)
5. \( q(x) = 3x^4 - 5x^3 + 10x + 12 \)
6. \( r(x) = 4x^5 - 2x + 18 \)
7. \( f(x) = x^7 - 6x^5 - 3x^4 + x^3 + 4x^2 - 120 \)
8. \( g(x) = 5x^6 - 3x^4 + 5x^3 + 2x^2 - 15 \)
9. \( h(x) = 6x^5 - 3x^4 + 12x^3 + 18x^2 - 9x + 21 \)
10. \( p(x) = 2x^7 - 3x^6 + 11x^5 - 20x^2 + 11 \)
Rational Zero Theorem

Find Rational Zeros

Example 1  Find all of the rational zeros of \( f(x) = 5x^3 + 12x^2 - 29x + 12 \).

From the corollary to the Fundamental Theorem of Algebra, we know that there are exactly 3 complex roots. According to Descartes’ Rule of Signs there are 2 or 0 positive real roots and 1 negative real root. The possible rational zeros are \( \pm 1, \pm 2, \pm 3, \pm 6, \pm 12, \pm \frac{1}{5}, \pm \frac{2}{5}, \pm \frac{3}{5}, \pm \frac{4}{5}, \pm \frac{12}{5} \). Make a table and test some possible rational zeros.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>5</th>
<th>12</th>
<th>-29</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>17</td>
<td>-12</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Since \( f(1) = 0 \), you know that \( x = 1 \) is a zero.

The depressed polynomial is \( 5x^2 + 17x - 12 \), which can be factored as \( (5x - 3)(x + 4) \).

By the Zero Product Property, this expression equals 0 when \( x = \frac{3}{5} \) or \( x = -4 \).

The rational zeros of this function are \( \frac{1}{2}, \frac{3}{5}, \text{ and } -4 \).

Example 2  Find all of the zeros of \( f(x) = 8x^4 + 2x^3 + 5x^2 + 2x - 3 \).

There are 4 complex roots, with 1 positive real root and 3 or 1 negative real roots. The possible rational zeros are \( \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{1}{8}, \pm \frac{3}{2}, \pm \frac{3}{4}, \text{ and } \pm \frac{3}{8} \).

Make a table and test some possible values.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>8</th>
<th>2</th>
<th>5</th>
<th>2</th>
<th>-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>10</td>
<td>15</td>
<td>17</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>18</td>
<td>41</td>
<td>84</td>
<td>165</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>8</td>
<td>6</td>
<td>8</td>
<td>6</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Since \( f\left(\frac{1}{2}\right) = 0 \), we know that \( x = \frac{1}{2} \) is a root.

The zeros of this function are \( \frac{1}{2}, -\frac{3}{4}, \text{ and } \pm i \).

Exercises

Find all of the rational zeros of each function.

1. \( f(x) = x^3 + 4x^2 - 25x - 28 \)
2. \( f(x) = x^3 + 6x^2 + 4x + 24 \)

Find all of the zeros of each function.

3. \( f(x) = x^4 + 2x^3 - 11x^2 + 8x - 60 \)
4. \( f(x) = 4x^4 + 5x^3 + 30x^2 + 45x - 54 \)
Rational Zero Theorem

List all of the possible rational zeros of each function.

1. \( h(x) = x^3 - 5x^2 + 2x + 12 \)  
2. \( s(x) = x^4 - 8x^3 + 7x - 14 \)

3. \( f(x) = 3x^5 - 5x^2 + x + 6 \)  
4. \( p(x) = 3x^2 + x + 7 \)

5. \( g(x) = 5x^3 + x^2 - x + 8 \)  
6. \( q(x) = 6x^8 + x^3 - 3 \)

Find all of the rational zeros of each function.

7. \( q(x) = x^3 + 3x^2 - 6x - 8 \)  
8. \( v(x) = x^3 - 9x^2 + 27x - 27 \)

9. \( c(x) = x^3 - x^2 - 8x + 12 \)  
10. \( f(x) = x^4 - 49x^2 \)

11. \( h(x) = x^3 - 7x^2 + 17x - 15 \)  
12. \( b(x) = x^3 + 6x + 20 \)

13. \( f(x) = x^3 - 6x^2 + 4x - 24 \)  
14. \( g(x) = 2x^3 + 3x^2 - 4x - 4 \)

15. \( h(x) = 2x^3 - 7x^2 - 21x + 54 \)  
16. \( z(x) = x^4 - 3x^3 + 5x^2 - 27x - 36 \)

17. \( d(x) = x^4 + x^3 + 16 \)  
18. \( n(x) = x^4 - 2x^3 - 3 \)

19. \( p(x) = 2x^4 - 7x^3 + 4x^2 + 7x - 6 \)  
20. \( q(x) = 6x^4 - 9x^3 + 40x^2 + 7x - 12 \)

Find all of the zeros of each function.

21. \( f(x) = 2x^4 + 7x^3 - 2x^2 - 19x - 12 \)  
22. \( q(x) = x^4 - 4x^3 + x^2 + 16x - 20 \)

23. \( h(x) = x^6 - 8x^3 \)  
24. \( g(x) = x^6 - 1 \)

25. **TRAVEL** The height of a box that Joan is shipping is 3 inches less than the width of the box. The length is 2 inches more than twice the width. The volume of the box is 1540 in\(^3\). What are the dimensions of the box?

26. **GEOMETRY** The height of a square pyramid is 3 meters shorter than the side of its base. If the volume of the pyramid is 432 m\(^3\), how tall is it? Use the formula \( V = \frac{1}{3} Bh \).
6-8 Word Problem Practice

Rational Zero Theorem

1. ROOTS Paul was examining an old algebra book. He came upon a page about polynomial equations and saw the polynomial below.

\[ x^4 + 8 \]

As you can see, all the middle terms were blotted out by an ink spill. What are all the possible rational roots of this polynomial?

2. IRRATIONAL CONSTANTS Cherie was given a polynomial whose constant term was \( \sqrt{2} \). Is it possible for this polynomial to have a rational root? If it is not, explain why not. If it is possible, give an example of such a polynomial with a rational root.

3. MARKOV CHAINS Tara is a mathematician who specializes in probability. In the course of her work, she needed to find the roots of the polynomial

\[ p(x) = 288x^4 - 288x^3 + 106x^2 - 17x + 1. \]

What are the roots of \( p(x) \)?

4. PYRAMIDS The Great Pyramid in Giza, Egypt has a square base with side lengths of 5\( x \) yards and a height of 4\( x - 50 \) yards. The volume of the Great Pyramid is 3,125,000 cubic yards. Use a calculator to find the value of \( x \) and the dimensions of the pyramid.

5. BOXES Devon made a box with length \( x + 1 \), width \( x + 3 \), and height \( x - 3 \).

a. What is the volume of Devon’s box as a function of \( x \)?

b. What is \( x \) if the volume of the box is equal to 1001 cubic inches?

c. What is \( x \) if the volume of the box is equal to \( 14 \frac{5}{8} \) cubic inches?
7-1 Study Guide

Operations on Functions

Arithmetic Operations

Operations with Functions

<table>
<thead>
<tr>
<th>Operation</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum</td>
<td>((f + g)(x) = f(x) + g(x))</td>
</tr>
<tr>
<td>Difference</td>
<td>((f - g)(x) = f(x) - g(x))</td>
</tr>
<tr>
<td>Product</td>
<td>((f \cdot g)(x) = f(x) \cdot g(x))</td>
</tr>
<tr>
<td>Quotient</td>
<td>(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \cdot g(x) \neq 0)</td>
</tr>
</tbody>
</table>

**Example**

Find \((f + g)(x)\), \((f - g)(x)\), \((f \cdot g)(x)\), and \(\left(\frac{f}{g}\right)(x)\) for \(f(x) = x^2 + 3x - 4\) and \(g(x) = 3x - 2\).

\[
\begin{align*}
(f + g)(x) &= f(x) + g(x) \\
&= (x^2 + 3x - 4) + (3x - 2) \\
&= x^2 + 6x - 6
\end{align*}
\]

Addition of functions

\[
\begin{align*}
(f - g)(x) &= f(x) - g(x) \\
&= (x^2 + 3x - 4) - (3x - 2) \\
&= x^2 - 2
\end{align*}
\]

Subtraction of functions

\[
\begin{align*}
(f \cdot g)(x) &= f(x) \cdot g(x) \\
&= (x^2 + 3x - 4)(3x - 2) \\
&= x^2(3x - 2) + 3x(3x - 2) - 4(3x - 2) \\
&= 3x^3 - 2x^2 + 9x^2 - 6x - 12x + 8 \\
&= 3x^3 + 7x^2 - 18x + 8
\end{align*}
\]

Multiplication of functions

\[
\begin{align*}
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \\
&= \frac{x^2 + 3x - 4}{3x - 2}, \ x \neq \frac{2}{3}
\end{align*}
\]

Division of functions

\[
\begin{align*}
(f(x) = x^2 + 3x - 4 \text{ and } g(x) = 3x - 2)
\end{align*}
\]

**Exercises**

Find \((f + g)(x)\), \((f - g)(x)\), \((f \cdot g)(x)\), and \(\left(\frac{f}{g}\right)(x)\) for each \(f(x)\) and \(g(x)\).

1. \(f(x) = 8x - 3; g(x) = 4x + 5\)
2. \(f(x) = x^2 + x - 6; g(x) = x - 2\)

3. \(f(x) = 3x^2 - x + 5; g(x) = 2x - 3\)
4. \(f(x) = 2x - 1; g(x) = 3x^2 + 11x - 4\)

5. \(f(x) = x^2 - 1; g(x) = \frac{1}{x + 1}\)
Composition of Functions Suppose \( f \) and \( g \) are functions such that the range of \( g \) is a subset of the domain of \( f \). Then the composite function \( f \circ g \) can be described by the equation \( (f \circ g)(x) = f(g(x)) \).

Example 1 For \( f = \{(1, 2), (3, 3), (2, 4), (4, 1)\} \) and \( g = \{(1, 3), (3, 4), (2, 2), (4, 1)\} \), find \( f \circ g \) and \( g \circ f \) if they exist.

\[
\begin{align*}
  f(g(1)) &= f(3) = 3 & f(g(2)) &= f(2) = 4 \\
g(f(1)) &= g(3) = 1 & f(g(4)) &= f(4) = 1 & f(4) &= f(1) = 2, \\
g(f(2)) &= g(2) = 2 & g(f(3)) &= g(3) = 4 & g(4) &= g(1) = 3, \\
g(f(3)) &= g(4) = 1 & & & \\
g(f(4)) &= g(2) = 2 & & & \\
\end{align*}
\]

So \( f \circ g = \{(1, 3), (2, 4), (3, 1), (4, 2)\} \)

Example 2 Find \( [g \circ h](x) \) and \( [h \circ g](x) \) for \( g(x) = 3x - 4 \) and \( h(x) = x^2 - 1 \).

\[
\begin{align*}
  [g \circ h](x) &= g[h(x)] \\
  &= g(x^2 - 1) \\
  &= 3(x^2 - 1) - 4 \\
  &= 3x^2 - 7 \\
  [h \circ g](x) &= h[g(x)] \\
  &= h(3x - 4) \\
  &= (3x - 4)^2 - 1 \\
  &= 9x^2 - 24x + 16 - 1 \\
  &= 9x^2 - 24x + 15 \\
\end{align*}
\]

Exercises

For each pair of functions, find \( f \circ g \) and \( g \circ f \), if they exist.

1. \( f = \{(-1, 2), (5, 6), (0, 9)\} \), \( g = \{(6, 0), (2, -1), (9, 5)\} \)
2. \( f = \{(5, -2), (9, 8), (-4, 3), (0, 4)\} \), \( g = \{(3, 7), (-2, 6), (4, -2), (8, 10)\} \)

Find \( [f \circ g](x) \) and \( [g \circ f](x) \), if they exist.

3. \( f(x) = 2x + 7; g(x) = -5x - 1 \)
4. \( f(x) = x^2 - 1; g(x) = -4x^2 \)

5. \( f(x) = x^2 + 2x; g(x) = x - 9 \)
6. \( f(x) = 5x + 4; g(x) = 3 - x \)
7-1 Practice

Operations on Functions

Find \((f + g)(x), (f - g)(x), (f \cdot g)(x),\) and \(\left(\frac{f}{g}\right)(x)\) for each \(f(x)\) and \(g(x)\).

1. \(f(x) = 2x + 1\)
   \[g(x) = x - 3\]

2. \(f(x) = 8x^2\)
   \[g(x) = \frac{1}{x^3}\]

3. \(f(x) = x^2 + 7x + 12\)
   \[g(x) = x^2 - 9\]

For each pair of functions, find \(f \circ g\) and \(g \circ f\), if they exist.

4. \(f = \{(−9, −1), (−1, 0), (3, 4)\}\)
   \[g = \{(0, −9), (−1, 3), (4, −1)\}\]

5. \(f = \{(−4, 3), (0, −2), (1, −2)\}\)
   \[g = \{(−2, 0), (3, 1)\}\]

6. \(f = \{(−4, −5), (0, 3), (1, 6)\}\)
   \[g = \{(6, 1), (−5, 0), (3, −4)\}\]

7. \(f = \{(0, −3), (1, −3), (6, 8)\}\)
   \[g = \{(8, 2), (−3, 0), (−3, 1)\}\]

Find \([g \circ h](x)\) and \([h \circ g](x)\), if they exist.

8. \(g(x) = 3x\)
   \[h(x) = x - 4\]

9. \(g(x) = −8x\)
   \[h(x) = 2x + 3\]

10. \(g(x) = x + 6\)
    \[h(x) = 3x^2\]

11. \(g(x) = x + 3\)
    \[h(x) = 2x^2\]

12. \(g(x) = −2x\)
    \[h(x) = x^2 + 3x + 2\]

13. \(g(x) = x - 2\)
    \[h(x) = 3x^2 + 1\]

If \(f(x) = x^2, g(x) = 5x,\) and \(h(x) = x + 4\), find each value.

14. \(f[g(1)]\)

15. \(g[h(-2)]\)

16. \(h[f(4)]\)

17. \(f[h(-9)]\)

18. \(h[g(-3)]\)

19. \(g[f(8)]\)

20. BUSINESS The function \(f(x) = 1000 - 0.01x^2\) models the manufacturing cost per item when \(x\) items are produced, and \(g(x) = 150 - 0.001x^2\) models the service cost per item. Write a function \(C(x)\) for the total manufacturing and service cost per item.

21. MEASUREMENT The formula \(f = \frac{n}{12}\) converts inches \(n\) to feet \(f\), and \(m = \frac{f}{5280}\) converts feet to miles \(m\). Write a composition of functions that converts inches to miles.
1. **AREA** Bernard wants to know the area of a figure made by joining an equilateral triangle and square along an edge. The function \( f(s) = \frac{\sqrt{3}}{4}s^2 \) gives the area of an equilateral triangle with side \( s \). The function \( g(s) = s^2 \) gives the area of a square with side \( s \). What function \( h(s) \) gives the area of the figure as a function of its side length \( s \)?

2. **PRICING** A computer company decides to continuously adjust the pricing of and discounts to its products in an effort to remain competitive. The function \( P(t) \) gives the sale price of its Super2000 computer as a function of time. The function \( D(t) \) gives the value of a special discount it offers to valued customers. How much would valued customers have to pay for one Super2000 computer?

3. **LAVA** The temperature of lava has been measured at up to 2000°F. A freshly ejected lava rock immediately begins to cool down. The temperature of the lava rock in degrees Fahrenheit as a function of time is given by \( T(t) \). Let \( C(F) \) be the function that gives degrees Celsius as a function of degrees Fahrenheit. What function gives the temperature of the lava rock in degrees Celsius as a function of time?

4. **ENGINEERING** A group of engineers is designing a staple gun. One team determines that the speed of impact \( s \) of the staple (in feet per second) as a function of the handle length \( \ell \) (in inches) is given by \( s(\ell) = 40 + 3\ell \). A second team determines that the number of sheets \( N \) that can be stapled as a function of the impact speed is given by \( N(s) = \frac{s - 10}{3} \). What function gives \( N \) as a function of \( \ell \)?

5. **HOT AIR BALLOONS** Hannah and Terry went on a one-hour hot air balloon ride. Let \( T(A) \) be the outside air temperature as a function of altitude and let \( A(t) \) be the altitude of the balloon as a function of time.

   a. What function describes the air temperature Hannah and Terry felt at different times during their trip?

   b. Sketch a graph of the function you wrote for part a based on the graphs for \( T(A) \) and \( A(t) \) that are given.
Inverse Functions and Relations

<table>
<thead>
<tr>
<th>Inverse Relations</th>
<th>Two relations are inverse relations if and only if whenever one relation contains the element ((a, b)), the other relation contains the element ((b, a)).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Property of Inverse Functions</td>
<td>Suppose (f) and (f^{-1}) are inverse functions. Then (f(a) = b) if and only if (f^{-1}(b) = a).</td>
</tr>
</tbody>
</table>

**Example**

Find the inverse of the function \(f(x) = \frac{2}{5}x - \frac{1}{5}\). Then graph the function and its inverse.

**Step 1** Replace \(f(x)\) with \(y\) in the original equation.

\[ f(x) = \frac{2}{5}x - \frac{1}{5} \rightarrow y = \frac{2}{5}x - \frac{1}{5} \]

**Step 2** Interchange \(x\) and \(y\).

\[ x = \frac{2}{5}y - \frac{1}{5} \]

**Step 3** Solve for \(y\).

\[
\begin{align*}
5x + 1 &= 2y \\
\frac{1}{2}(5x + 1) &= y
\end{align*}
\]

The inverse of \(f(x) = \frac{2}{5}x - \frac{1}{5}\) is \(f^{-1}(x) = \frac{1}{2}(5x + 1)\).

**Exercises**

Find the inverse of each function. Then graph the function and its inverse.

1. \(f(x) = \frac{2}{3}x - 1\)
2. \(f(x) = 2x - 3\)
3. \(f(x) = \frac{1}{4}x - 2\)
Inverse Functions and Relations

Verifying Inverses

**Inverse Functions**

Two functions \( f(x) \) and \( g(x) \) are inverse functions if and only if \( [f \circ g](x) = x \) and \( [g \circ f](x) = x \).

**Example 1**

Determine whether \( f(x) = 2x - 7 \) and \( g(x) = \frac{1}{2}(x + 7) \) are inverse functions.

\[
[f \circ g](x) = f[g(x)] = f\left[\frac{1}{2}(x + 7)\right] = 2\left[\frac{1}{2}(x + 7)\right] - 7 = x + 7 - 7 = x = x
\]

\[g \circ f](x) = g[f(x)] = \frac{1}{2}(2x - 7 + 7) = \frac{1}{2}(2x) = x = x
\]

The functions are inverses since both \([f \circ g](x) = x\) and \([g \circ f](x) = x\).

**Example 2**

Determine whether \( f(x) = 4x + \frac{1}{3} \) and \( g(x) = \frac{1}{4}x - 3 \) are inverse functions.

\[
[f \circ g](x) = f\left[\frac{1}{4}x - 3\right] = 4\left[\frac{1}{4}x - 3\right] + \frac{1}{3} = x - 12 + \frac{1}{3} = x - 11\frac{2}{3}
\]

Since \([f \circ g](x) \neq x\), the functions are not inverses.

**Exercises**

Determine whether each pair of functions are inverse functions. Write yes or no.

1. \( f(x) = 3x - 1 \) \( g(x) = \frac{1}{3}x + \frac{1}{3} \)
2. \( f(x) = \frac{1}{4}x + 5 \) \( g(x) = 4x - 20 \)
3. \( f(x) = \frac{1}{2}x - 10 \) \( g(x) = 2x + \frac{1}{10} \)
4. \( f(x) = 2x + 5 \) \( g(x) = 5x + 2 \)
5. \( f(x) = 8x - 12 \) \( g(x) = \frac{1}{8}x + 12 \)
6. \( f(x) = -2x + 3 \) \( g(x) = -\frac{1}{2}x + \frac{3}{2} \)
7. \( f(x) = 4x - \frac{1}{2} \) \( g(x) = \frac{1}{4}x + \frac{1}{8} \)
8. \( f(x) = 2x - \frac{3}{5} \) \( g(x) = \frac{1}{10}(5x + 3) \)
9. \( f(x) = 4x + \frac{1}{2} \) \( g(x) = \frac{1}{2}x - \frac{3}{2} \)
10. \( f(x) = 10 - \frac{x}{2} \) \( g(x) = 20 - 2x \)
11. \( f(x) = 4x - \frac{4}{5} \) \( g(x) = \frac{x}{4} + \frac{1}{5} \)
12. \( f(x) = 9 + \frac{3}{2}x \) \( g(x) = \frac{2}{3}x - 6 \)
7-2 Practice

Inverse Functions and Relations

Find the inverse of each relation.

1. \{(0, 3), (4, 2), (5, -6)\}

2. \{(-5, 1), (-5, -1), (-5, 8)\}

3. \{(-3, -7), (0, -1), (5, 9), (7, 13)\}

4. \{(8, -2), (10, 5), (12, 6), (14, 7)\}

5. \{(-5, -4), (1, 2), (3, 4), (7, 8)\}

6. \{(-3, 9), (-2, 4), (0, 0), (1, 1)\}

Find the inverse of each function. Then graph the function and its inverse.

7. \(f(x) = \frac{3}{4}x\)

8. \(g(x) = 3 + x\)

9. \(y = 3x - 2\)

10. \(f(x) = x + 6\)  
     \(g(x) = x - 6\)

11. \(f(x) = -4x + 1\)  
     \(g(x) = \frac{1}{4}(1 - x)\)

12. \(g(x) = 13x - 13\)  
     \(h(x) = \frac{1}{13}x - 1\)

13. \(f(x) = 2x\)  
     \(g(x) = -2x\)

14. \(f(x) = \frac{6}{7}x\)  
     \(g(x) = \frac{7}{6}x\)

15. \(g(x) = 2x - 8\)  
     \(h(x) = \frac{1}{2}x + 4\)

16. MEASUREMENT The points (63, 121), (71, 180), (67, 140), (65, 108), and (72, 165) give the weight in pounds as a function of height in inches for 5 students in a class. Give the points for these students that represent height as a function of weight.

17. REMODELING The Clearys are replacing the flooring in their 15 foot by 18 foot kitchen. The new flooring costs $17.99 per square yard. The formula \(f(x) = 9x\) converts square yards to square feet.

a. Find the inverse \(f^{-1}(x)\). What is the significance of \(f^{-1}(x)\) for the Clearys?

b. What will the new flooring cost the Clearys?
1. VOLUME Jason wants to make a spherical water cooler that can hold half a cubic meter of water. He knows that 
\[ V = \frac{4}{3}\pi r^3, \]
but he needs to know how to find \( r \) given \( V \). Find this inverse function.

2. EXERCISE Alex began a new exercise routine. To gain the maximum benefit from his exercise, Alex calculated his maximum target heart rate using the function 
\[ f(x) = 0.85(220 - x), \]
where \( x \) represents his age. Find the inverse of this function.

3. ROCKETS The altitude of a rocket in feet as a function of time is given by 
\[ f(t) = 49t^2, \]
where \( t \geq 0 \). Find the inverse of this function and determine the times when the rocket will be 10, 100, and 1000 feet high. Round your answers to the nearest hundredth of a second.

4. SELF-INVERTIBLE Karen finds the incomplete graph of a function in the back of her engineering handbook. The function is graphed in the figure below.

Karen knows that this function is its own inverse. Armed with this knowledge, extend the graph for values of \( x \) between \(-7\) and 2.

5. PLANETS The approximate distance of a planet from the Sun is given by 
\[ d = T^\frac{2}{3}, \]
where \( d \) is distance in astronomical units and \( T \) is the period of its orbit in Earth years. An astronomical unit is the distance between Earth and the Sun.

a. Solve for \( T \) in terms of \( d \).

b. Pluto is about 39.44 times as far from the Sun as Earth. About how many years does it take Pluto to orbit the Sun?
Square Root Functions

A function that contains the square root of a variable expression is a square root function. The domain of a square root function is those values for which the radicand is greater than or equal to 0.

Example

Graph \( y = \sqrt{3x - 2} \). State its domain and range.

Since the radicand cannot be negative, the domain of the function is \( 3x - 2 \geq 0 \) or \( x \geq \frac{2}{3} \).

The \( x \)-intercept is \( \frac{2}{3} \). The range is \( y \geq 0 \).

Make a table of values and graph the function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{2}{3} )</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3 ( \sqrt{7} )</td>
<td></td>
</tr>
</tbody>
</table>

Exercises

Graph each function. State the domain and range.

1. \( y = \sqrt{2x} \)
2. \( y = -3\sqrt{x} \)
3. \( y = -\sqrt{\frac{x}{2}} \)
4. \( y = 2\sqrt{x - 3} \)
5. \( y = -\sqrt{2x - 3} \)
6. \( y = \sqrt{2x + 5} \)
Square Root Inequalities

A square root inequality is an inequality that contains the square root of a variable expression. Use what you know about graphing square root functions and graphing inequalities to graph square root inequalities.

Example

Graph \( y \leq \sqrt{2x - 1} + 2 \).

Graph the related equation \( y = \sqrt{2x - 1} + 2 \). Since the boundary should be included, the graph should be solid.

The domain includes values for \( x \geq \frac{1}{2} \), so the graph is to the right of \( x = \frac{1}{2} \).

Exercises

Graph each inequality.

1. \( y < 2\sqrt{x} \)

2. \( y > \sqrt{x + 3} \)

3. \( y < 3\sqrt{2x - 1} \)

4. \( y < \sqrt{3x - 4} \)

5. \( y \geq \sqrt{x + 1} - 4 \)

6. \( y > 2\sqrt{2x - 3} \)

7. \( y \geq \sqrt{3x + 1} - 2 \)

8. \( y \leq \sqrt{4x - 2} + 1 \)

9. \( y < 2\sqrt{2x - 1} - 4 \)
Graph each function. State the domain and range.

1. \( y = \sqrt{5x} \)

2. \( y = -\sqrt{x - 1} \)

3. \( y = 2\sqrt{x + 2} \)

4. \( y = \sqrt{3x - 4} \)

5. \( y = \sqrt{x + 7} - 4 \)

6. \( y = 1 - \sqrt{2x + 3} \)

Graph each inequality.

7. \( y \geq -\sqrt{6x} \)

8. \( y \leq \sqrt{x - 5} + 3 \)

9. \( y > -2\sqrt{3x + 2} \)

10. **ROLLER COASTERS** The velocity of a roller coaster as it moves down a hill is \( v = \sqrt{v_0^2 + 64h} \), where \( v_0 \) is the initial velocity and \( h \) is the vertical drop in feet. If \( v = 70 \) feet per second and \( v_0 = 8 \) feet per second, find \( h \).

11. **WEIGHT** Use the formula \( d = \sqrt{\frac{3960^2 W_e}{W_s}} - 3960 \), which relates distance from Earth \( d \) in miles to weight. If an astronaut’s weight on Earth \( W_e \) is 148 pounds and in space \( W_s \) is 115 pounds, how far from Earth is the astronaut?
1. SQUARES  Cathy is building a square roof for her garage. The roof will occupy 625 square feet. What are the dimensions of the roof?

2. PENDULUMS  The period of a pendulum, or the time it takes to complete one swing, is given by the formula

\[ p = 2\pi \sqrt{\frac{L}{g}} \]

where \( L \) is the length in meters of the pendulum and \( g \) is acceleration due to gravity, 9.8 m/s\(^2\). Find the period of a pendulum that is 0.65 meters long. Round to the nearest tenth.

3. REFLEXES  Rachel and Ashley are testing one another’s reflexes. Rachel drops a ruler from a given height so that it falls between Ashley’s thumb and index finger. Ashley tries to catch the ruler before it falls through her hand. The time required to catch the ruler is given by \( t = \sqrt{\frac{d}{4}} \) where \( d \) is measured in feet. Complete the table. Round your answers to the nearest hundredth.

<table>
<thead>
<tr>
<th>Distance (in.)</th>
<th>Reflex Time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

4. DISTANCE  Lance is standing at the side of a road watching a cyclist go by. The distance between Lance and the cyclist as a function of time is given by \( d = \sqrt{9 + 36t^2} \). Graph this function. Find the distance between Lance and the cyclist after 3 seconds.

5. STARS  The intensity of the light from an object varies inversely with the square of the distance. In other words, \( I = \frac{k}{d^2} \).

a. Solve the equation to find \( d \) in terms of \( I \).

b. The stars Antares and Spica have the same apparent magnitudes. However, their absolute magnitudes, or intensities, differ. Let \( I_1 \) and \( I_2 \) be their absolute magnitudes and let \( d_1 \) and \( d_2 \) be their respective distances from Earth. What is the ratio of \( d_2 \) to \( d_1 \)?
Rational Exponents and Radicals

Write $28^{\frac{1}{2}}$ in radical form.

Notice that $28 > 0$.

$$28^{\frac{1}{2}} = \sqrt{28}$$
$$= \sqrt{2^2 \cdot 7}$$
$$= 2\sqrt{7}$$

Example 2 Evaluate $\left(\frac{-8}{-125}\right)^{\frac{1}{3}}$.

Notice that $-8 < 0$, $-125 < 0$, and 3 is odd.

$$\left(\frac{-8}{-125}\right)^{\frac{1}{3}} = \left(\frac{\sqrt[3]{-8}}{\sqrt[3]{-125}}\right)$$
$$= \frac{-2}{-5}$$
$$= \frac{2}{5}$$

Exercises

Write each expression in radical form, or write each radical in exponential form.

1. $11^{\frac{1}{2}}$

2. $15^{\frac{1}{3}}$

3. $300^{\frac{3}{2}}$

4. $\sqrt[4]{47}$

5. $\sqrt[3]{3a^5b^2}$

6. $\sqrt[4]{162p^5}$

Evaluate each expression.

7. $-27^{\frac{2}{3}}$

8. $216^{\frac{1}{2}}$

9. $(0.0004)^{\frac{1}{2}}$
Rational Exponents

Simplify Expressions  All the properties of powers from Lesson 6-1 apply to rational exponents. When you simplify expressions with rational exponents, leave the exponent in rational form, and write the expression with all positive exponents. Any exponents in the denominator must be positive integers.

When you simplify radical expressions, you may use rational exponents to simplify, but your answer should be in radical form. Use the smallest index possible.

Example 1  Simplify \( y^{\frac{2}{3}} \cdot y^{\frac{3}{8}} \).

\[
y^{\frac{2}{3}} \cdot y^{\frac{3}{8}} = y^{\frac{2}{3} + \frac{3}{8}} = y^{\frac{16}{24} + \frac{9}{24}} = y^{\frac{25}{24}}
\]

Example 2  Simplify \( \sqrt[4]{144x^6} \).

\[
\sqrt[4]{144x^6} = (144x^6)^{\frac{1}{4}}
= (2^4 \cdot 3^2 \cdot x^6)^{\frac{1}{4}}
= (2^{1\cdot4} \cdot 3^{\frac{2}{4}} \cdot x^{6\cdot\frac{1}{4}})
= 2 \cdot 3^{\frac{1}{2}} \cdot x^{\frac{3}{4}}
= 2x \sqrt[4]{3x^3}
\]

Exercises

Simplify each expression.

1. \( x^{\frac{3}{5}} \cdot x^{\frac{6}{5}} \)
2. \( (y^{\frac{3}{4}})^{\frac{3}{2}} \)
3. \( p^{\frac{4}{5}} \cdot p^{\frac{7}{10}} \)

4. \( (m^{\frac{2}{3}})^{\frac{3}{2}} \)
5. \( x^{\frac{3}{8}} \cdot x^{\frac{4}{3}} \)
6. \( (s^{\frac{1}{4}})^{\frac{3}{2}} \)

7. \( \frac{p}{p^{\frac{1}{3}}} \)
8. \( \frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}} \)
9. \( \sqrt[6]{128} \)

10. \( \sqrt[4]{49} \)
11. \( \sqrt[5]{288} \)
12. \( \sqrt[3]{32} \cdot 3\sqrt[4]{16} \)

13. \( \sqrt[3]{25} \cdot \sqrt[4]{125} \)
14. \( \sqrt[6]{16} \)
15. \( \frac{a\sqrt{b^4}}{\sqrt[3]{ab^2}} \)
### 7-6 Practice

**Rational Exponents**

Write each expression in radical form, or write each radical in exponential form.

1. \(5^{\frac{1}{3}}\)  
2. \(6^{\frac{2}{5}}\)  
3. \(m^{\frac{4}{7}}\)  
4. \((n^3)^{\frac{2}{5}}\)

5. \(\sqrt{79}\)  
6. \(\sqrt[4]{153}\)  
7. \(3\sqrt[3]{27m^6n^4}\)  
8. \(\sqrt[5]{2a^{10}b}\)

Evaluate each expression.

9. \(81^{\frac{1}{4}}\)  
10. \(1024^{\frac{1}{5}}\)  
11. \(8^{\frac{5}{3}}\)

12. \(-256^{\frac{3}{4}}\)  
13. \((-64)^{\frac{2}{3}}\)  
14. \(27^{\frac{1}{3}} \cdot 27^{\frac{4}{3}}\)

15. \(\left(\frac{125}{216}\right)^{\frac{2}{3}}\)  
16. \(\frac{64\frac{3}{2}}{343\frac{3}{5}}\)  
17. \(\left(25^{\frac{1}{2}}\right)\left(-64^{\frac{1}{3}}\right)\)

Simplify each expression.

18. \(g^{\frac{4}{7}} \cdot g^{\frac{3}{7}}\)  
19. \(s^{\frac{3}{4}} \cdot s^{\frac{13}{4}}\)  
20. \(\left(u^{\frac{1}{3}}\right)^{\frac{5}{3}}\)  
21. \(y^{-\frac{1}{2}}\)

22. \(b^{\frac{3}{5}}\)  
23. \(q^{\frac{3}{5}} \div q^{\frac{3}{5}}\)  
24. \(\frac{t^{\frac{2}{3}}}{5t^{\frac{1}{2}} \cdot t^{-\frac{3}{4}}}\)  
25. \(\frac{2z^{\frac{1}{2}}}{z^{\frac{1}{2}} - 1}\)

26. \(\sqrt[10]{85}\)  
27. \(\sqrt{12} \cdot \sqrt[3]{12}\)  
28. \(\sqrt[4]{6} \cdot 3\sqrt[4]{6}\)  
29. \(\frac{a}{\sqrt{3b}}\)

30. **ELECTRICITY** The amount of current in amps \(I\) that an appliance uses can be calculated using the formula \(I = \left(\frac{P}{R}\right)^{\frac{1}{2}}\), where \(P\) is the power in watts and \(R\) is the resistance in ohms. How much current does an appliance use if \(P = 500\) watts and \(R = 10\) ohms? Round your answer to the nearest tenth.

31. **BUSINESS** A company that produces DVDs uses the formula \(C = 88n^{\frac{1}{3}} + 330\) to calculate the cost \(C\) in dollars of producing \(n\) DVDs per day. What is the company’s cost to produce 150 DVDs per day? Round your answer to the nearest dollar.
1. **SQUARING THE CUBE** A cube has side length \( s \). What side length of the square will cause its area to have the same numerical value as the volume of the cube? Write your answer using rational exponents.

2. **WATER TOWER** Typically, drinking water for towns is stored in water towers. A water tower in Edmond, Oklahoma is 218 feet high and holds half a million gallons. One town is replacing its water tower. Residents of the town insist that their new tower be a sphere. If the new tank will hold 10 times as much water as the old tank, how many times longer should the radius of the new tank be compared to the old tank? Write your answer using rational exponents.

3. **BALLOONS** A spherical balloon is being inflated faster and faster. The volume of the balloon as a function of time is \( 9\pi t^2 \). What is the radius of the balloon as a function of time? Write your answer using rational exponents.

4. **INTEREST** Rita opened a bank account that accumulated interest at the rate of 1% compounded annually. Her money accumulated interest in that account for 8 years. She then took all of her money out of that account and placed it into another account that paid 5% interest compounded annually. After 4 years, she took all of her money out of that account. What single interest rate when compounded annually would give her the same outcome for those 12 years? Round your answer to the nearest hundredth of a percent.

5. **CELLS** The number of cells in a cell culture grows exponentially. The number of cells in the culture as a function of time is given by the expression \( N(6)^{\frac{t}{5}} \), where \( t \) is measured in hours and \( N \) is the initial size of the culture.

   a. After 3 hours, there were 1728 cells in the culture. What is \( N \)?

   b. How many cells were in the culture after 20 minutes? Express your answer in simplest form.

   c. How many cells were in the culture after 2.5 hours? Express your answer in simplest form.
Graphing Exponential Functions

Exponential Growth  An exponential growth function has the form \( y = b^x \), where \( b > 1 \). The graphs of exponential equations can be transformed by changing the value of the constants \( a, h, \) and \( k \) in the exponential equation: \( f(x) = ab^{x-h} + k \).

<table>
<thead>
<tr>
<th>Parent Function of Exponential Growth Functions, ( f(x) = b^x, b &gt; 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. The function is continuous, one-to-one, and increasing.</td>
</tr>
<tr>
<td>2. The domain is the set of all real numbers.</td>
</tr>
<tr>
<td>3. The ( x )-axis is the asymptote of the graph.</td>
</tr>
<tr>
<td>4. The range is the set of all non-zero real numbers.</td>
</tr>
<tr>
<td>5. The graph contains the point ((0, 1)).</td>
</tr>
</tbody>
</table>

Example  Graph \( y = 4^x + 2 \). State the domain and range.

Make a table of values. Connect the points to form a smooth curve.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>2.25</td>
<td>3</td>
<td>6</td>
<td>18</td>
<td>66</td>
</tr>
</tbody>
</table>

The domain of the function is all real numbers, while the range is the set of all positive real numbers greater than 2.

Exercises  Graph each function. State the domain and range.

1. \( y = 3(2)^x \)

2. \( y = \frac{1}{3} (3)^x \)

3. \( y = 0.25(5)^x \)

4. \( y = 2(3)^x \)

5. \( y = 4^x - 2 \)

6. \( y = 2^{x+5} \)
Graphing Exponential Functions

Exponential Decay The following table summarizes the characteristics of exponential decay functions.

| Parent Function of Exponential Decay Functions, \( f(x) = b^x, 0 < b < 1 \) | 1. The function is continuous, one-to-one, and decreasing.  
2. The domain is the set of all real numbers.  
3. The x-axis is the asymptote of the graph.  
4. The range is the set of all positive real numbers.  
5. The graph contains the point \((0, 1)\). |

Example 1 Graph \( y = \left(\frac{1}{2}\right)^x \). State the domain and range.

Make a table of values. Connect the points to form a smooth curve. The domain is all real numbers and the range is the set of all positive real numbers.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0.5</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Exercises

Graph each function. State the domain and range.

1. \( y = 6\left(\frac{1}{2}\right)^x \)
2. \( y = -2\left(\frac{1}{4}\right)^x \)
3. \( y = -0.4(0.2)^x \)

4. \( y = \left(\frac{2}{5}\right)\left(\frac{1}{2}\right)^{x-1} + 2 \)
5. \( y = 4\left(\frac{1}{5}\right)^x + 3 - 1 \)
6. \( y = \left(-\frac{1}{3}\right)\left(\frac{3}{4}\right)^{x-5} + 6 \)
Graphing Exponential Functions

Graph each function. State the domain and range.

1. \( y = 1.5(2)^x \)

2. \( y = 4(3)^x \)

3. \( y = 3(0.5)^x \)

4. \( y = 5\left(\frac{1}{2}\right)^x - 8 \)

5. \( y = -2\left(\frac{1}{4}\right)^x - 3 \)

6. \( y = \frac{1}{2}(3)^{x+4} - 5 \)

7. **BIOLOGY** The initial number of bacteria in a culture is 12,000. The culture doubles each day.
   a. Write an exponential function to model the population \( y \) of bacteria after \( x \) days.
   b. How many bacteria are there after 6 days?

8. **EDUCATION** A college with a graduating class of 4000 students in the year 2008 predicts that its graduating class will grow 5% per year. Write an exponential function to model the number of students \( y \) in the graduating class \( t \) years after 2008.
1. **GOLF BALLS** A golf ball manufacturer packs 3 golf balls into a single package. Three of these packages make a gift box. Three gift boxes make a value pack. The display shelf is high enough to stack 3 value packs one on top of the other. Three such columns of value packs make up a display front. Three display fronts can be packed in a single shipping box and shipped to various retail stores. How many golf balls are in a single shipping box?

2. **FOLDING** Paper thickness ranges from 0.0032 inch to 0.0175 inch. Kay folds a piece of paper 0.01 inch thick in half over and over until it is at least 25 layers thick. How many times does she fold the paper in half and how many layers are there? How thick is the folded paper?

3. **SUBSCRIPTIONS** Subscriptions to an online arts and crafts club have been increasing by 20% every year. The club began with 40 members.

<table>
<thead>
<tr>
<th>Year</th>
<th>Subscriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>40</td>
</tr>
<tr>
<td>1</td>
<td>48</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Make a graph of the number of subscribers over the first 5 years of the club’s existence.

4. **TENNIS SHOES** The cost of a pair of tennis shoes increases about 5.1% every year. About how much would a $50 pair of tennis shoes cost 25 years from now?

5. **MONEY** Sam opened a savings account that compounds interest at a rate of 3% annually. Let $P$ be the initial amount Sam deposited and let $t$ be the number of years the account has been open.

   a. Write an equation to find $A$, the amount of money in the account after $t$ years. Assume that Sam made no more additional deposits and no withdrawals.

   b. If Sam opened the account with $500 and made no deposits or withdrawals, how much is in the account 10 years later?

   c. What is the least number of years it would take for such an account to double in value?
Solving Exponential Equations and Inequalities

Solve Exponential Equations  All the properties of rational exponents that you know also apply to real exponents. Remember that \( a^m \cdot a^n = a^{m+n}, (a^m)^n = a^{mn}, \) and \( a^m \div a^n = a^{m-n}. \)

**Property of Equality for Exponential Functions**

If \( b \) is a positive number other than 1, then \( b^x = b^y \) if and only if \( x = y. \)

**Example 1**

\[ 4^{x - 1} = 2^{x + 5}. \]

Original equation

Rewrite 4 as \( 2^2. \)

\[ 2(x - 1) = x + 5 \]

Prop. of Inequality for Exponential Functions

\[ 2x - 2 = x + 5 \]

Distributive Property

\[ x = 7 \]

Subtract \( x \) and add 2 to each side.

**Example 2**

Write an exponential function whose graph passes through the points \((0, 3)\) and \((4, 81)\).

The \( y \)-intercept is \((0, 3)\), so \( a = 3. \) Since the other point is \((4, 81)\), \( b = \sqrt[4]{81} = \sqrt{27} \approx 2.280. \)

Simplifying \( \sqrt[4]{27} \approx 2.280, \) the equation is \( y = 3(2.280)^x. \)

**Exercises**

Solve each equation.

1. \( 3^{2x - 1} = 3^{x + 2} \)
2. \( 2^{3x} = 4^{x + 2} \)
3. \( 3^{2x - 1} = \frac{1}{9} \)
4. \( 4^{x + 1} = 8^{2x + 3} \)
5. \( 8^{x - 2} = \frac{1}{16} \)
6. \( 25^{2x} = 125^{x + 2} \)
7. \( 9^{x + 1} = 27^{x + 4} \)
8. \( 36^{2x + 4} = 216^{x + 5} \)
9. \( \left(\frac{1}{64}\right)^{x - 2} = 16^{3x + 1} \)

Write an exponential function for the graph that passes through the given points.

10. \((0, 4)\) and \((2, 36)\)
11. \((0, 6)\) and \((1, 81)\)
12. \((0, 5)\) and \((6, 320)\)
13. \((0, 2)\) and \((5, 486)\)
14. \((0, 8)\) and \(\left(3, \frac{27}{8}\right)\)
15. \((0, 1)\) and \((4, 625)\)
16. \((0, 3)\) and \((3, 24)\)
17. \((0, 12)\) and \((4, 144)\)
18. \((0, 9)\) and \((2, 49)\)
Solve Exponential Inequalities

An exponential inequality is an inequality involving exponential functions.

| Property of Inequality for Exponential Functions | If $b > 1$
|-------------------------------------------------|--------------------------------------------------|
|                                                 | then $b^x > b^y$ if and only if $x > y$
|                                                 | and $b^x < b^y$ if and only if $x < y$. |

Example

Solve $5^{2x-1} > \frac{1}{125}$.

- $5^{2x-1} > \frac{1}{125}$: Original inequality
- $5^{2x-1} > 5^{-3}$: Rewrite $\frac{1}{125}$ as $5^{-3}$.

2. $2x - 1 > -3$: Prop. of Inequality for Exponential Functions

- $2x > -2$: Add 1 to each side.
- $x > -1$: Divide each side by 2.

The solution set is $\{x \mid x > -1\}$.

Exercises

Solve each inequality.

1. $3^x - 4 < \frac{1}{27}$
2. $4^{2x - 2} > 2x + 1$
3. $5^x < 125^{x - 5}$
4. $10^{4x + 1} > 100^{x - 2}$
5. $7^x < 49^{1 - x}$
6. $8^{2x - 5} < 4^x + 8$
7. $16 \geq 4^{x + 5}$
8. $\left(\frac{1}{27}\right)^{2x + 1} \leq \left(\frac{1}{243}\right)^{3x - 2}$
9. $\left(\frac{1}{2}\right)^{x - 3} > 8^{2x}$
10. $\frac{1}{81} < 9^{2x - 4}$
11. $32^{3x - 4} > 128^{4x + 3}$
12. $27^{2x - 5} < \left(\frac{1}{9}\right)^{5x}$
13. $\left(\frac{1}{25}\right)^{2x - 1} \leq 125^{3x + 1}$
14. $\left(\frac{7}{343}\right)^{x - 3} \geq \left(\frac{1}{49}\right)^{2x + 1}$
15. $\left(\frac{9}{27}\right)^{6x - 1} \geq \left(\frac{27}{9}\right)^{-x + 6}$
Solving Exponential Equations and Inequalities

Solve each equation.

1. \(4^x + 35 = 64^{x - 3}\)
2. \(\left(\frac{1}{64}\right)^{0.5x - 3} = 8^{9x - 2}\)

3. \(3^x - 4 = 9^x + 28\)
4. \(\left(\frac{1}{4}\right)^{2x + 2} = 64^{x - 1}\)

5. \(\left(\frac{1}{2}\right)^x - 3 = 16^{2x + 1}\)
6. \(3^{6x - 2} = \left(\frac{1}{9}\right)^x + 1\)

Write an exponential function for the graph that passes through the given points.

7. (0, 5) and (4, 3125)
8. (0, 8) and (4, 2048)
9. (0, \(\frac{3}{4}\)) and (2, 36.75)

10. (0, -0.2) and (-3, -3.125)
11. (0, 15) and \(\left(\frac{2}{15}\right)\frac{16}{15}\)
12. (0, 0.7) and \(\left(\frac{1}{2}\right)^{3.5}\)

Solve each inequality.

13. \(400 > \left(\frac{1}{20}\right)^{7x + 8}\)
14. \(10^{2x + 7} \geq 1000^x\)
15. \(\left(\frac{1}{16}\right)^{3x - 4} \leq 64^{x - 1}\)

16. \(\left(\frac{1}{8}\right)^{x - 6} < 4^{4x + 5}\)
17. \(\left(\frac{1}{36}\right)^x + 8 \leq 216^{x - 3}\)
18. \(128^{x + 3} < \left(\frac{1}{1024}\right)^{2x}\)

19. At time \(t\), there are \(216^t + 18\) bacteria of type A and \(36^{2x + 8}\) bacteria of type B organisms in a sample. When will the number of each type of bacteria be equal?
1. **BANKING** The certificate of deposit that Siobhan bought on her birthday pays interest according to the formula 
   \[ A = 1200 \left(1 + \frac{0.052}{12}\right)^{48} \]. What is the annual interest rate?

2. **INTEREST** Marty invested $2000 in an account that pays at least 4% annual interest. He wants to see how much money he will have over the next few years. Graph the inequality 
   \[ y \geq 2000(1 + 0.04)^x \] to show his potential earnings.

3. **BUSINESS** Ahmed’s consulting firm began with 23 clients. After 7 years, he now has 393 clients. Write an exponential equation describing the firm’s growth.

4. **POPULATION** In 2000, the world population was calculated to be 6,071,675,206. In 2008, it was 6,679,493,893. Write an exponential equation to model the growth of the world population over these 8 years. Round the base to the nearest thousandth.
   
   **Source:** U.S. Census Bureau

5. **BUSINESS** Ingrid and Alberto each opened a business in 2000. Ingrid started with 2 employees and in 2003 she had 50 employees. Alberto began with 32 employees and in 2007 he had 310 employees. Since 2000, each company has experienced exponential growth.
   
   a. Write an exponential equation representing the growth for each business.
   
   b. Calculate the number of employees each company had in 2005.
   
   c. Is it reasonable to expect that a business can experience exponential growth? Explain your answer.
Logarithmic Functions and Expressions

### Definition of Logarithm with Base $b$

Let $b$ and $x$ be positive numbers, $b \neq 1$. The logarithm of $x$ with base $b$ is denoted $\log_b(x)$ and is defined as the exponent $y$ that makes the equation $b^y = x$ true.

The inverse of the exponential function $y = b^x$ is the logarithmic function $x = b^y$.

This function is usually written as $y = \log_b(x)$.

#### Example 1

Write an exponential equation equivalent to $\log_3(243) = 5$.

$3^5 = 243$

#### Example 2

Write a logarithmic equation equivalent to $6^{-3} = \frac{1}{216}$.

$log_6\left(\frac{1}{216}\right) = -3$

#### Example 3

Evaluate $\log_8(16)$.

$8^{\frac{4}{3}} = 16$, so $\log_8(16) = \frac{4}{3}$.

### Exercises

Write each equation in exponential form.

1. $\log_{15}(225) = 2$  
2. $\log_3\left(\frac{1}{27}\right) = -3$  
3. $\log_4(32) = \frac{5}{2}$  

Write each equation in logarithmic form.

4. $2^7 = 128$  
5. $3^{-4} = \frac{1}{81}$  
6. $\left(\frac{1}{7}\right)^3 = \frac{1}{343}$  

7. $7^{-2} = \frac{1}{49}$  
8. $2^9 = 512$  
9. $64^{\frac{2}{3}} = 16$  

Evaluate each expression.

10. $\log_4(64)$  
11. $\log_2(64)$  
12. $\log_{100}(100,000)$  

13. $\log_5(625)$  
14. $\log_{27}(81)$  
15. $\log_{26}(5)$  

16. $\log_2\left(\frac{1}{128}\right)$  
17. $\log_{10}(0.0001)$  
18. $\log_4\left(\frac{1}{32}\right)$
Graphing Logarithmic Functions

The function $y = \log_b x$, where $b \neq 1$, is called a logarithmic function. The graph of $f(x) = \log_b x$ represents a parent graph of the logarithmic functions. Properties of the parent function are described in the following table.

| Parent function of Logarithmic Functions, $f(x) = \log_b x$ | 1. The function is continuous and one-to-one.  
2. The domain is the set of all positive real numbers.  
3. The $y$-axis is an asymptote of the graph.  
4. The range is the set of all real numbers.  
5. The graph contains the point (1, 0). |

The graphs of logarithmic functions can be transformed by changing the value of the constants $a$, $h$, and $k$ in the equation $f(x) = a \log_b (x - h) + k$.

Example

Graph $f(x) = -3 \log_{10} (x - 2) + 1$.

This is a transformation of the graph of $f(x) = \log_{10} x$.

- $|a| = 3$: The graph expands vertically.
- $a < 0$: The graph is reflected across the $x$-axis.
- $h = 2$: The graph is translated 2 units to the right.
- $k = 1$: The graph is translated 1 unit up.

Exercises

Graph each function.

1. $f(x) = 4 \log_2 x$
2. $f(x) = 4 \log_4 (x - 1)$
3. $f(x) = 2 \log_4 (x + 3) - 2$
Write each equation in exponential form.

1. \( \log_6 216 = 3 \)  
2. \( \log_2 64 = 6 \)  
3. \( \log_3 \frac{1}{81} = -4 \)  
4. \( \log_{10} 0.00001 = -5 \)  
5. \( \log_{25} 5 = \frac{1}{2} \)  
6. \( \log_{32} 8 = \frac{3}{5} \)

Write each equation in logarithmic form.

7. \( 5^3 = 125 \)  
8. \( 7^0 = 1 \)  
9. \( 3^4 = 81 \)  
10. \( 3^{-4} = \frac{1}{81} \)  
11. \( \left(\frac{1}{4}\right)^3 = \frac{1}{64} \)  
12. \( 7776^{\frac{1}{3}} = 6 \)

Evaluate each expression.

13. \( \log_3 81 \)  
14. \( \log_{10} 0.0001 \)  
15. \( \log_2 \frac{1}{16} \)  
16. \( \log_{\frac{1}{3}} 27 \)

17. \( \log_9 1 \)  
18. \( \log_8 4 \)  
19. \( \log_7 \frac{1}{49} \)  
20. \( \log_6 6^4 \)

Graph each function.

21. \( f(x) = \log_2 (x - 2) \)  
22. \( f(x) = -2 \log_4 x \)

23. **SOUND** An equation for loudness, in decibels, is \( L = 10 \log_{10} R \), where \( R \) is the relative intensity of the sound. Sounds that reach levels of 120 decibels or more are painful to humans. What is the relative intensity of 120 decibels?

24. **INVESTING** Maria invests $1000 in a savings account that pays 4% interest compounded annually. The value of the account \( A \) at the end of five years can be determined from the equation \( \log_{10} A = \log_{10} [1000(1 + 0.04)^5] \). Write this equation in exponential form.
1. CHEMISTRY  The pH of a solution is found by the formula \( pH = -\log H \), where \( H \) stands for the hydrogen ion concentration in the formula. What is the pH of a solution to the nearest hundredth when \( H \) is 1356?

2. FIND THE ERROR  Michio wanted to find the value of \( x \) in the equation \( 2(3)^x = 34 \). He first converted the equation to \( \log_3 2x = 17 \). Next he wrote \( 2x = 3^{17} \) and used a calculator to find \( x = 64,570,081 \). Was his answer correct? If not, what was his mistake and what is the right answer?

3. SOUND  The decibel level \( L \) of a sound is determined by the formula \( L = 10 \log_{10} \frac{I}{M} \). Find \( I \) in terms of \( M \) for a noise with a decibel level of 120.

4. EARTHQUAKES  The intensity of an earthquake can be measured on the Richter scale using the formula \( y = 10^{R - 1} \), where \( y \) is the absolute intensity of the earthquake and \( R \) is its Richter scale measurement.

<table>
<thead>
<tr>
<th>Richter Scale Number</th>
<th>Absolute Intensity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
</tr>
<tr>
<td>4</td>
<td>1000</td>
</tr>
<tr>
<td>5</td>
<td>10,000</td>
</tr>
</tbody>
</table>

An earthquake in San Francisco in 1906 had an absolute intensity of 6,000,000. What was that earthquake’s measurement on the Richter scale?

5. GAMES  Julio and Natalia decided to play a game in which they each selected a logarithmic function and compare their functions to see which gave larger values. Julio selected the function \( f(x) = 10 \log_2 x \) and Natalia selected the function \( 2 \log_{10} x \).

a. Which of the functions has a larger value when \( x = 7 \)?

b. Which of their functions has a larger value when \( x = 1 \)?

c. Do you think the base or the multiplier is more important in determining the value of a logarithmic function?
8-4 Study Guide

Solving Logarithmic Equations and Inequalities

Solving Logarithmic Equations

<table>
<thead>
<tr>
<th>Property of Equality for Logarithmic Functions</th>
<th>If $b$ is a positive number other than 1, then $\log_b x = \log_b y$ if and only if $x = y$.</th>
</tr>
</thead>
</table>

**Example 1** Solve $\log_2 2x = 3$.

1. $\log_2 2x = 3$ \hspace{1cm} \text{Original equation}
2. $2x = 2^3$ \hspace{1cm} \text{Definition of logarithm}
3. $2x = 8$ \hspace{1cm} \text{Simplify.}
4. $x = 4$ \hspace{1cm} \text{Simplify.}

The solution is $x = 4$.

**Example 2** Solve the equation $\log_2 (x + 17) = \log_2 (3x + 23)$.

Since the bases of the logarithms are equal, $(x + 17)$ must equal $(3x + 23)$.

1. $(x + 17) = (3x + 23)$
2. $-6 = 2x$
3. $x = -3$

**Exercises**

Solve each equation.

1. $\log_2 32 = 3x$
2. $\log_3 2c = -2$
3. $\log_{2x} 16 = -2$
4. $\log_{25} \left(\frac{x}{2}\right) = \frac{1}{2}$
5. $\log_4 (5x + 1) = 2$
6. $\log_8 (x - 5) = \frac{2}{3}$
7. $\log_4 (3x - 1) = \log_4 (2x + 3)$
8. $\log_2 (x^2 - 6) = \log_2 (2x + 2)$
9. $\log_x + 4 27 = 3$
10. $\log_2 (x + 3) = 4$
11. $\log_x 1000 = 3$
12. $\log_8 (4x + 4) = 2$
13. $\log_2 x = \log_2 12$
14. $\log_3 (x - 5) = \log_3 13$
15. $\log_{10} x = \log_{10} (5x - 20)$
16. $\log_5 x = \log_5 (2x - 1)$
17. $\log_4 (x+12) = \log_4 4x$
18. $\log_6 (x - 3) = \log_6 2x$
Solving Logarithmic Equations and Inequalities

Solving Logarithmic Inequalities

<table>
<thead>
<tr>
<th>Property of Inequality for Logarithmic Functions</th>
<th>If $b &gt; 1$, $x &gt; 0$, and $\log_b x &gt; y$, then $x &gt; b^y$.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>If $b &gt; 1$, $x &gt; 0$, and $\log_b x &lt; y$, then $0 &lt; x &lt; b^y$.</td>
</tr>
<tr>
<td></td>
<td>If $b &gt; 1$, then $\log_b x &gt; \log_b y$ if and only if $x &gt; y$, and $\log_b x &lt; \log_b y$ if and only if $x &lt; y$.</td>
</tr>
</tbody>
</table>

Example 1  Solve $\log_5 (4x - 3) < 3$.

$\log_5 (4x - 3) < 3$  Original equation

$0 < 4x - 3 < 5^3$  Property of Inequality

$3 < 4x < 125 + 3$  Simplify.

$3\frac{1}{4} < x < 32$  Simplify.

The solution set is $\left\{ x \mid \frac{3}{4} < x < 32 \right\}$.

Example 2  Solve the inequality $\log_3 (3x - 4) < \log_3 (x + 1)$.

Since the base of the logarithms are equal to or greater than 1, $3x - 4 < x + 1$.

$2x < 5$

$x < \frac{5}{2}$

Since $3x - 4$ and $x + 1$ must both be positive numbers, solve $3x - 4 = 0$ for the lower bound of the inequality.

The solution is $\left\{ x \mid \frac{4}{3} < x < \frac{5}{2} \right\}$.

Exercises

Solve each inequality.

1. $\log_2 2x > 2$
2. $\log_3 x > 2$
3. $\log_2 (3x + 1) < 4$
4. $\log_4 2x > -\frac{1}{2}$
5. $\log_3 (x + 3) < 3$
6. $\log_{27} 6x > \frac{2}{3}$
7. $\log_{10} 5x < \log_{10} 30$
8. $\log_{10} x < \log_{10} (2x - 4)$
9. $\log_{10} 3x < \log_{10} (7x - 8)$
10. $\log_2 (8x + 5) > \log_2 (9x - 18)$
11. $\log_{10} (3x + 7) < \log_{10} (7x - 3)$
12. $\log_2 (3x - 4) < \log_2 (2x + 7)$
Solve each equation.

1. \(x + 5 = \log_4 256\) 
2. \(3x - 5 = \log_2 1024\)

3. \(\log_3 (4x - 17) = 5\) 
4. \(\log_5 (3 - x) = 5\)

5. \(\log_{13} (x^2 - 4) = \log_{13} 3x\) 
6. \(\log_3 (x - 5) = \log_3 (3x - 25)\)

Solve each inequality.

7. \(\log_9 (-6x) < 1\) 
8. \(\log_9 (x + 2) > \log_9 (6 - 3x)\)

9. \(\log_{11} (x + 7) < 1\) 
10. \(\log_{81} x \leq 0.75\)

11. \(\log_2 (x + 6) < \log_2 17\) 
12. \(\log_{12} (2x - 1) > \log_{12} (5x - 16)\)

13. \(\log_9 (2x - 1) < 0.5\) 
14. \(\log_{10} (x - 5) > \log_{10} 2x\)

15. \(\log_3 (x + 12) > \log_3 2x\) 
16. \(\log_3 (0.3x + 5) > \log_3 (x - 2)\)

17. \(\log_2 (x + 3) < \log_2 (1 - 3x)\) 
18. \(\log_6 (3 - x) \leq \log_6 (x - 1)\)

19. **WILDLIFE** An ecologist discovered that the population of a certain endangered species has been doubling every 12 years. When the population reaches 20 times the current level, it may no longer be endangered. Write the logarithmic expression that gives the number of years it will take for the population to reach that level.
8-4 Word Problem Practice

Solving Logarithmic Equations and Inequalities

1. **FISH** The population of silver carp has been growing in the Mississippi River. About every 3 years, the population doubles. Write logarithmic expression that gives the number of years it will take for the population to increase by a factor of ten.

2. **POWERS** Haley tries to solve the equation \( \log_4 2x = 5 \). She got the wrong answer. What was her mistake? What should the correct answer be?

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( \log_4 2x = 5 )</td>
</tr>
<tr>
<td>2.</td>
<td>( 2x = 4^5 )</td>
</tr>
<tr>
<td>3.</td>
<td>( x = 2^5 )</td>
</tr>
<tr>
<td>4.</td>
<td>( x = 32 )</td>
</tr>
</tbody>
</table>

3. **DIGITS** A computer programmer wants to write a formula that tells how many digits there are in a number \( n \), where \( n \) is a positive integer. For example, if \( n = 343 \), the formula should evaluate to 3 and if \( n = 10,000 \), the formula should evaluate to 5. Suppose \( 8 \leq \log_{10} n < 9 \). How many digits does \( n \) have?

4. **LOGARITHMS** Pauline knows that \( \log_b x = 3 \) and \( \log_b y = 5 \). She knows that this is the same as knowing that \( b^3 = x \) and \( b^5 = y \). Multiply these two equations together and rewrite it as an equation involving logarithms. What is \( \log_b xy \)?

5. **MUSIC** The first note on a piano keyboard corresponds to a pitch with a frequency of 27.5 cycles per second. With every successive note you go up the white and black keys of a piano, the pitch multiplies by a factor of \( \sqrt[12]{2} \). The formula for the frequency of the pitch sounded when the \( n \)th note up the keyboard is played is given by

\[
 n = 1 + 12 \log_{27.5} f .
\]

a. The pitch that orchestras tune to is the A above middle C. It has a frequency of 440 cycles per second. How many notes up the piano keyboard is this A?

b. Another pitch on the keyboard has a frequency of 1760 cycles per second. How many notes up the keyboard will this be found?
## Properties of Logarithms

Properties of exponents can be used to develop the following properties of logarithms.

<table>
<thead>
<tr>
<th>Property</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product Property of Logarithms</td>
<td>For all positive numbers $a$, $b$, and $x$, where $x \neq 1$, $\log_b ab = \log_b a + \log_b b$.</td>
</tr>
<tr>
<td>Quotient Property of Logarithms</td>
<td>For all positive numbers $a$, $b$, and $x$, where $x \neq 1$, $\log_b \frac{a}{b} = \log_b a - \log_b b$.</td>
</tr>
<tr>
<td>Power Property of Logarithms</td>
<td>For any real number $p$ and positive numbers $m$ and $b$, where $b \neq 1$, $\log_b m^p = p \log_b m$.</td>
</tr>
</tbody>
</table>

### Example

Use $\log_3 28 \approx 3.0331$ and $\log_3 4 \approx 1.2619$ to approximate the value of each expression.

**a.** $\log_3 36$

\[
\log_3 36 = \log_3 (3^2 \cdot 4) = \log_3 3^2 + \log_3 4 = 2 + \log_3 4 \approx 2 + 1.2619 \approx 3.2619
\]

**b.** $\log_3 7$

\[
\log_3 7 = \log_3 \left( \frac{28}{4} \right) = \log_3 28 - \log_3 4 \approx 3.0331 - 1.2619 \approx 1.7712
\]

**c.** $\log_3 256$

\[
\log_3 256 = \log_3 (4^4) = 4 \cdot \log_3 4 \approx 5.0476
\]

### Exercises

Use $\log_{12} 3 \approx 0.4421$ and $\log_{12} 7 \approx 0.7831$ to approximate the value of each expression.

1. $\log_{12} 21$

2. $\log_{12} \frac{7}{3}$

3. $\log_{12} 49$

4. $\log_{12} 36$

5. $\log_{12} 63$

6. $\log_{12} \frac{27}{49}$

7. $\log_{12} \frac{81}{49}$

8. $\log_{12} 16,807$

9. $\log_{12} 441$

Use $\log_5 3 \approx 0.6826$ and $\log_5 4 \approx 0.8614$ to approximate the value of each expression.

10. $\log_5 12$

11. $\log_5 100$

12. $\log_5 0.75$

13. $\log_5 144$

14. $\log_5 \frac{27}{16}$

15. $\log_5 375$

16. $\log_5 1.3$

17. $\log_5 \frac{9}{16}$

18. $\log_5 \frac{81}{5}$
Properties of Logarithms

Solve Logarithmic Equations  You can use the properties of logarithms to solve equations involving logarithms.

Example

Solve each equation.

a. \( \log_3 x - \log_3 4 = \log_3 25 \)

\[
\begin{align*}
2 \log_3 x - \log_3 4 &= \log_3 25 \\
\log_3 x^2 - \log_3 4 &= \log_3 25 \\
\log_3 \frac{x^2}{4} &= \log_3 25 \\
\frac{x^2}{4} &= 25 \\
x^2 &= 100 \\
x &= \pm 10
\end{align*}
\]

Since logarithms are undefined for \( x < 0 \), \(-10\) is an extraneous solution. The only solution is 10.

b. \( \log_2 x + \log_2 (x + 2) = 3 \)

\[
\log_2 x + \log_2 (x + 2) = 3 \quad \text{Original equation} \\
\log_2 x(x + 2) = 3 \quad \text{Product Property} \\
x(x + 2) = 2^3 \quad \text{Definition of logarithm} \\
x^2 + 2x = 8 \quad \text{Distributive Property} \\
x^2 + 2x - 8 = 0 \quad \text{Subtract 8 from each side.} \\
(x + 4)(x - 2) = 0 \quad \text{Factor.} \\
x = 2 \text{ or } x = -4 \quad \text{Zero Product Property}
\]

Since logarithms are undefined for \( x < 0 \), \(-4\) is an extraneous solution. The only solution is 2.

Exercises

Solve each equation. Check your solutions.

1. \( \log_5 4 + \log_5 2x = \log_5 24 \)  
2. \( 3 \log_4 6 - \log_4 8 = \log_4 x \)

3. \( \frac{1}{2} \log_6 25 + \log_6 x = \log_6 20 \)  
4. \( \log_2 4 - \log_2 (x + 3) = \log_2 8 \)

5. \( \log_6 2x - \log_6 3 = \log_6 (x - 1) \)  
6. \( 2 \log_4 (x + 1) = \log_4 (11 - x) \)

7. \( \log_2 x - 3 \log_2 5 = 2 \log_2 10 \)  
8. \( 3 \log_2 x - 2 \log_2 5x = 2 \)

9. \( \log_3 (c + 3) - \log_3 (4c - 1) = \log_3 5 \)  
10. \( \log_5 (x + 3) - \log_5 (2x - 1) = 2 \)
**8-5 Practice**

**Properties of Logarithms**

Use $\log_{10} 5 \approx 0.6990$ and $\log_{10} 7 \approx 0.8451$ to approximate the value of each expression.

1. $\log_{10} 35$
2. $\log_{10} 25$
3. $\log_{10} \frac{7}{5}$
4. $\log_{10} \frac{5}{7}$
5. $\log_{10} 245$
6. $\log_{10} 175$
7. $\log_{10} 0.2$
8. $\log_{10} \frac{25}{7}$

Solve each equation. Check your solutions.

9. $\log_{10} n = \frac{2}{3} \log_{10} 8$
10. $\log_{10} u = \frac{3}{2} \log_{10} 4$
11. $\log_{10} x + \log_{10} 9 = \log_{10} 54$
12. $\log_{10} 48 - \log_{10} w = \log_{10} 4$
13. $\log_{9} (3u + 14) - \log_{9} 5 = \log_{9} 2u$
14. $4 \log_{2} x + \log_{2} 5 = \log_{2} 405$
15. $\log_{3} y = -\log_{3} 16 + \frac{1}{3} \log_{3} 64$
16. $\log_{2} d = 5 \log_{2} 2 - \log_{2} 8$
17. $\log_{10} (3m - 5) + \log_{10} m = \log_{10} 2$
18. $\log_{10} (b + 3) + \log_{10} b = \log_{10} 4$
19. $\log_{8} (t + 10) - \log_{8} (t - 1) = \log_{8} 12$
20. $\log_{3} (a + 3) + \log_{3} (a + 2) = \log_{3} 6$
21. $\log_{10} (r + 4) - \log_{10} r = \log_{10} (r + 1)$
22. $\log_{4} (x^2 - 4) - \log_{4} (x + 2) = \log_{4} 1$
23. $\log_{10} 4 + \log_{10} w = 2$
24. $\log_{3} (n - 3) + \log_{3} (n + 4) = 1$
25. $3 \log_{5} (x^2 + 9) - 6 = 0$
26. $\log_{16} (9x + 5) - \log_{16} (x^2 - 1) = \frac{1}{2}$
27. $\log_{6} (2x - 5) + 1 = \log_{6} (7x + 10)$
28. $\log_{2} (5y + 2) - 1 = \log_{2} (1 - 2y)$
29. $\log_{10} (c^2 - 1) - 2 = \log_{10} (c + 1)$
30. $\log_{7} x + 2 \log_{7} x - \log_{7} 3 = \log_{7} 72$

31. **SOUND** Recall that the loudness $L$ of a sound in decibels is given by $L = 10 \log_{10} R$, where $R$ is the sound’s relative intensity. If the intensity of a certain sound is tripled, by how many decibels does the sound increase?

32. **EARTHQUAKES** An earthquake rated at 3.5 on the Richter scale is felt by many people, and an earthquake rated at 4.5 may cause local damage. The Richter scale magnitude reading $m$ is given by $m = \log_{10} x$, where $x$ represents the amplitude of the seismic wave causing ground motion. How many times greater is the amplitude of an earthquake that measures 4.5 on the Richter scale than one that measures 3.5?
1. **MENTAL COMPUTATION** Jessica has memorized \( \log_5 2 \approx 0.4307 \) and \( \log_5 3 \approx 0.6826 \). Using this information, to the nearest ten-thousandth, what power of 5 is equal to 6?

2. **POWERS** A chemist is testing a soft drink. The pH of a solution is given by \( -\log_{10} C \), where \( C \) is the concentration of hydrogen ions. The pH of a popular soft drink is 2.5. If the concentration of hydrogen ions is increased by a factor of 100, what is the new pH of the solution?

3. **LUCKY MATH** Frank is solving a problem involving logarithms. He does everything correctly except for one thing. He mistakenly writes \( \log_2 a + \log_2 b = \log_2 (a + b) \).

   However, after substituting the values for \( a \) and \( b \) in his problem, he amazingly still gets the right answer! The value of \( a \) was 11. What must the value of \( b \) have been?

4. **LENGTHS** Charles has two poles. One pole has length equal to \( \log_7 21 \) and the other has length equal to \( \log_7 25 \). Express the length of both poles joined end to end as the logarithm of a single number.

5. **SIZE** Alicia wanted to try to quantify the terms tiny, small, medium, large, big, huge, and humongous. She picked a number of objects and classified them with these adjectives of size. She noticed that the scale seemed exponential. Therefore, she came up with the following definition. Define \( S \) to be \( \frac{1}{3} \log_3 V \), where \( V \) is volume in cubic feet. Then use the following table to find the appropriate adjective.

<table>
<thead>
<tr>
<th>( S ) satisfies</th>
<th>Adjective</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2 \leq S &lt; -1)</td>
<td>tiny</td>
</tr>
<tr>
<td>(-1 \leq S &lt; 0)</td>
<td>small</td>
</tr>
<tr>
<td>(0 \leq S &lt; 1)</td>
<td>medium</td>
</tr>
<tr>
<td>(1 \leq S &lt; 2)</td>
<td>large</td>
</tr>
<tr>
<td>(2 \leq S &lt; 3)</td>
<td>big</td>
</tr>
<tr>
<td>(3 \leq S &lt; 4)</td>
<td>huge</td>
</tr>
<tr>
<td>(4 \leq S &lt; 5)</td>
<td>humongous</td>
</tr>
</tbody>
</table>

   a. Derive an expression for \( S \) applied to a cube in terms of \( \ell \) where \( \ell \) is the side length of a cube.

   b. How many cubes, each one foot on a side, would have to be put together to get an object that Alicia would call “big”?

   c. How likely is it that a large object attached to a big object would result in a huge object, according to Alicia’s scale?
Common Logarithms

Base 10 logarithms are called common logarithms. The expression \( \log_{10} x \) is usually written without the subscript as \( \log x \). Use the \( \text{LOG} \) key on your calculator to evaluate common logarithms.

The relation between exponents and logarithms gives the following identity.

| Inverse Property of Logarithms and Exponents | \( 10^{\log x} = x \) |

**Example 1** Evaluate \( \log 50 \) to the nearest ten-thousandth.

Use the \( \text{LOG} \) key on your calculator. To four decimal places, \( \log 50 \approx 1.6990 \).

**Example 2** Solve \( 3^{2x+1} = 12 \).

\[
\begin{align*}
3^{2x+1} &= 12 & \text{Original equation} \\
\log 3^{2x+1} &= \log 12 & \text{Logarithmic property} \\
(2x + 1) \log 3 &= \log 12 & \text{Power property of logarithms} \\
2x + 1 &= \frac{\log 12}{\log 3} & \text{Divide each side by } \log 3 \text{.} \\
2x &= \frac{\log 12}{\log 3} - 1 & \text{Subtract 1 from each side.} \\
x &= \frac{1}{2} \left( \frac{\log 12}{\log 3} - 1 \right) & \text{Multiply each side by } \frac{1}{2} \text{.} \\
x &\approx \frac{1}{2} \left( \frac{1.0792}{0.4771} - 1 \right) & \text{Use a calculator.} \\
x &\approx 0.6309 
\end{align*}
\]

**Exercises**

Use a calculator to evaluate each expression to the nearest ten-thousandth.

1. \( \log 18 \) 
2. \( \log 39 \) 
3. \( \log 120 \)

4. \( \log 5.8 \) 
5. \( \log 42.3 \) 
6. \( \log 0.003 \)

Solve each equation or inequality. Round to the nearest ten-thousandth.

7. \( 4^x = 12 \) 
8. \( 6^x + 2 = 18 \)

9. \( 5^{4x - 2} = 120 \) 
10. \( 7^{3x - 1} \geq 21 \)

11. \( 2.4^x + 4 = 30 \) 
12. \( 6.5^x \geq 200 \)

13. \( 3.6^{4x - 1} = 85.4 \) 
14. \( 2^x + 5 = 3^x - 2 \)

15. \( 9^x = 4^{5x + 2} \) 
16. \( 6^x - 5 = 2^{5x + 3} \)
Change of Base Formula

The following formula is used to change expressions with different logarithmic bases to common logarithm expressions.

\[
\log_a n = \frac{\log_b n}{\log_b a}
\]

Express \( \log_8 15 \) in terms of common logarithms. Then round to the nearest ten-thousandth.

\[
\log_8 15 = \frac{\log_{10} 15}{\log_{10} 8}
\]

\[
\approx 1.3023
\]

Simplify.

The value of \( \log_8 15 \) is approximately 1.3023.

**Exercises**

Express each logarithm in terms of common logarithms. Then approximate its value to the nearest ten-thousandth.

1. \( \log_3 16 \)
2. \( \log_2 40 \)
3. \( \log_5 35 \)
4. \( \log_4 22 \)
5. \( \log_{12} 200 \)
6. \( \log_2 50 \)
7. \( \log_5 0.4 \)
8. \( \log_3 2 \)
9. \( \log_4 28.5 \)
10. \( \log_3 (20)^2 \)
11. \( \log_6 (5)^4 \)
12. \( \log_8 (4)^5 \)
13. \( \log_5 (8)^3 \)
14. \( \log_2 (3.6)^6 \)
15. \( \log_{12} (10.5)^4 \)
16. \( \log_3 \sqrt{150} \)
17. \( \log_4 \sqrt{39} \)
18. \( \log_5 \sqrt{1600} \)
Use a calculator to evaluate each expression to the nearest ten-thousandth.

1. \( \log 101 \)
2. \( \log 2.2 \)
3. \( \log 0.05 \)

Use the formula \( \text{pH} = -\log [H^+] \) to find the pH of each substance given its concentration of hydrogen ions. Round to the nearest tenth.

4. milk: \([H^+] = 2.51 \times 10^{-7} \) mole per liter
5. acid rain: \([H^+] = 2.51 \times 10^{-6} \) mole per liter
6. black coffee: \([H^+] = 1.0 \times 10^{-5} \) mole per liter
7. milk of magnesia: \([H^+] = 3.16 \times 10^{-11} \) mole per liter

Solve each equation or inequality. Round to the nearest ten-thousandth.

8. \( 2^x < 25 \)
9. \( 5^a = 120 \)
10. \( 6^y = 45.6 \)
11. \( 9^n \geq 100 \)
12. \( 3.5^x = 47.9 \)
13. \( 8.2^z = 64.5 \)
14. \( 2^{x+1} \leq 7.31 \)
15. \( 4^{2x} = 27 \)
16. \( 2^{x-4} = 82.1 \)
17. \( 9^{x-2} > 38 \)
18. \( 5^{x+3} = 17 \)
19. \( 30^2 = 50 \)
20. \( 5^{x^2-3} = 72 \)
21. \( 4^{2x} = 9^{x+1} \)
22. \( 2^{n+1} = 5^{2n-1} \)

Express each logarithm in terms of common logarithms. Then approximate its value to the nearest ten-thousandth.

23. \( \log_{10} 12 \)
24. \( \log_{10} 32 \)
25. \( \log_{11} 9 \)
26. \( \log_{2} 18 \)
27. \( \log_{9} 6 \)
28. \( \log_{7} \sqrt{8} \)

29. **HORTICULTURE** Siberian irises flourish when the concentration of hydrogen ions \([H^+]\) in the soil is not less than \(1.58 \times 10^{-8} \) mole per liter. What is the pH of the soil in which these irises will flourish?

30. **ACIDITY** The pH of vinegar is 2.9 and the pH of milk is 6.6. Approximately how many times greater is the hydrogen ion concentration of vinegar than of milk?

31. **BIOLOGY** There are initially 1000 bacteria in a culture. The number of bacteria doubles each hour. The number of bacteria \(N\) present after \(t\) hours is \(N = 1000(2)^t\). How long will it take the culture to increase to 50,000 bacteria?

32. **SOUND** An equation for loudness \(L\) in decibels is given by \(L = 10 \log R\), where \(R\) is the sound's relative intensity. An air-raid siren can reach 150 decibels and jet engine noise can reach 120 decibels. How many times greater is the relative intensity of the air-raid siren than that of the jet engine noise?
1. OTHER BASES Jamie needs to figure out what $\log_2 3$ is, but she only has a table of common logarithms. In the table, she finds that $\log_{10} 2 \approx 0.3010$ and $\log_{10} 3 \approx 0.4771$. Using this information, to the nearest thousandth, what is $\log_2 3$?

2. pH The pH of a solution is given by $-\log_{10} C$, where $C$ is the concentration of hydrogen ions in moles per liter. A solution of baking soda creates a hydrogen ion concentration $5 \times 10^{-9}$ of mole per liter. What is the pH of a solution of baking soda? Round your answer to the nearest tenth.

3. GRAPHING The graph of $y = \log_{10} x$ is shown below. Use the fact that $\frac{1}{\log_{10} 2} \approx 3.32$ to sketch a graph of $y = \log_2 x$ on the same graph.

4. SCIENTIFIC NOTATION When a number $n$ is written in scientific notation, it has the form $n = s \times 10^p$, where $s$ is a number greater than or equal to 1 and less than 10 and $p$ is an integer. Show that $p \leq \log_{10} n < p + 1$.

5. LOG TABLE Marjorie is looking through some old science books owned by her grandfather. At the back of one of them, there is a table of logarithms base 10. However, the book is worn out and some of the entries are unreadable.

Table of Common Logarithms

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\log_{10} x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.3010</td>
</tr>
<tr>
<td>3</td>
<td>0.4771</td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.6990</td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

a. Approximately what are the missing entries in the table? Round off your answers to the nearest thousandth.

b. How can you use this table to determine $\log_{10} 1.5$?
Base e and Natural Logarithms

The irrational number \( e \approx 2.71828 \ldots \) often occurs as the base for exponential and logarithmic functions that describe real-world phenomena.

The functions \( f(x) = e^x \) and \( f(x) = \ln x \) are inverse functions.

Natural base expressions can be evaluated using the \( e^x \) and \( \ln \) keys on your calculator.

**Example 1**
Write a logarithmic equation equivalent to \( e^{2x} = 7 \).

\[
e^{2x} = 7 \rightarrow \log_e 7 = 2x
\]

\[
2x = \ln 7
\]

**Example 2**
Write each logarithmic equation in exponential form.

a. \( \ln x \approx 0.3345 \)

\[
\ln x \approx 0.3345 \rightarrow \log_e x \approx 0.3345
\]

\[
x \approx e^{0.3345}
\]

b. \( \ln 42 = x \)

\[
\ln 42 = x \rightarrow \log_e 42 = x
\]

\[
42 = e^x
\]

**Exercises**

Write an equivalent exponential or logarithmic equation.

1. \( e^{15} = x \)
2. \( e^{3x} = 45 \)
3. \( \ln 20 = x \)
4. \( \ln x = 8 \)
5. \( e^{-5x} = 0.2 \)
6. \( \ln (4x) = 9.6 \)
7. \( e^{8.2} = 10x \)
8. \( \ln 0.0002 = x \)

Evaluate each logarithm to the nearest ten-thousandth.

9. \( \ln 12,492 \)
10. \( \ln 50.69 \)
11. \( \ln 9275 \)
12. \( \ln 0.835 \)
13. \( \ln 943 - \ln 181 \)
14. \( \ln 67 + \ln 103 \)
15. \( \ln 931 \cdot \ln 32 \)
16. \( \ln (139 - 45) \)
Equations and Inequalities with \( e \) and \( \ln \)

All properties of logarithms from earlier lessons can be used to solve equations and inequalities with natural logarithms.

### Example

Solve each equation or inequality.

a. \( 3e^{2x} + 2 = 10 \)

\[
3e^{2x} + 2 = 10 \quad \text{Original equation}
\]
\[
3e^{2x} = 8 \quad \text{Subtract 2 from each side.}
\]
\[
e^{2x} = \frac{8}{3} \quad \text{Divide each side by 3.}
\]
\[
\ln e^{2x} = \ln \left( \frac{8}{3} \right) \quad \text{Property of Equality for Logarithms}
\]
\[
2x = \frac{1}{2} \ln \frac{8}{3} \quad \text{Inverse Property of Exponents and Logarithms}
\]
\[
x = \frac{1}{2} \ln \frac{8}{3} \quad \text{Multiply each side by} \ \frac{1}{2}.
\]
\[
x \approx 0.4904 \quad \text{Use a calculator.}
\]

b. \( \ln (4x - 1) < 2 \)

\[
\ln (4x - 1) < 2 \quad \text{Original inequality}
\]
\[
e^{\ln (4x - 1)} < e^2 \quad \text{Write each side using exponents and base} \ e.
\]
\[
0 < 4x - 1 < e^2 \quad \text{Inverse Property of Exponents and Logarithms}
\]
\[
1 < 4x < e^2 + 1 \quad \text{Addition Property of Inequalities}
\]
\[
\frac{1}{4} < x < \frac{1}{4}(e^2 + 1) \quad \text{Multiplication Property of Inequalities}
\]
\[
0.25 < x < 2.0973 \quad \text{Use a calculator.}
\]

### Exercises

Solve each equation or inequality. Round to the nearest ten-thousandth.

1. \( e^{4x} = 120 \)
2. \( e^x \leq 25 \)
3. \( e^{x-2} + 4 = 21 \)

4. \( \ln 6x \geq 4 \)
5. \( \ln (x + 3) - 5 = -2 \)
6. \( e^{-3x} \leq 50 \)

7. \( e^{4x-1} - 3 = 12 \)
8. \( \ln (5x + 3) = 3.6 \)
9. \( 2e^{3x} + 5 = 2 \)

10. \( 6 + 3e^{x+1} = 21 \)
11. \( \ln (2x - 5) = 8 \)
12. \( \ln 5x + \ln 3x > 9 \)
Write an equivalent exponential or logarithmic equation.

1. \( \ln 50 = x \)  
2. \( \ln 36 = 2x \)  
3. \( \ln 6 \approx 1.7918 \)  
4. \( \ln 9.3 \approx 2.2300 \)

5. \( e^x = 8 \)  
6. \( e^5 = 10x \)  
7. \( e^{-x} = 4 \)  
8. \( e^2 = x + 1 \)

Solve each equation or inequality. Round to four decimal places.

9. \( e^x < 9 \)  
10. \( e^{-x} = 31 \)  
11. \( e^x = 1.1 \)  
12. \( e^x = 5.8 \)

13. \( 2e^x - 3 = 1 \)  
14. \( 5e^x + 1 \geq 7 \)  
15. \( 4 + e^x = 19 \)  
16. \( -3e^x + 10 < 8 \)

17. \( e^{3x} = 8 \)  
18. \( e^{-4x} = 5 \)  
19. \( e^{0.5x} = 6 \)  
20. \( 2e^{5x} = 24 \)

21. \( e^{2x} + 1 = 55 \)  
22. \( e^{3x} - 5 = 32 \)  
23. \( 9 + e^{2x} = 10 \)  
24. \( e^{-3x} + 7 \geq 15 \)

25. \( \ln 4x = 3 \)  
26. \( \ln (-2x) = 7 \)  
27. \( \ln 2.5x = 10 \)  
28. \( \ln (x - 6) = 1 \)

29. \( \ln (x + 2) = 3 \)  
30. \( \ln (x + 3) = 5 \)  
31. \( \ln 3x + \ln 2x = 9 \)  
32. \( \ln 5x + \ln x = 7 \)

33. **INVESTING** Sarita deposits $1000 in an account paying 3.4% annual interest compounded continuously. Use the formula for continuously compounded interest, \( A = Pe^{rt} \), where \( P \) is the principal, \( r \) is the annual interest rate, and \( t \) is the time in years.

   a. What is the balance in Sarita’s account after 5 years?

   b. How long will it take the balance in Sarita’s account to reach $2000?

34. **RADIOACTIVE DECAY** The amount of a radioactive substance \( y \) that remains after \( t \) years is given by the equation \( y = ae^{kt} \), where \( a \) is the initial amount present and \( k \) is the decay constant for the radioactive substance. If \( a = 100 \), \( y = 50 \), and \( k = -0.035 \), find \( t \).
Word Problem Practice

Base e and Natural Logarithms

1. INTEREST Horatio opens a bank account that pays 2.3% annual interest compounded continuously. He makes an initial deposit of 10,000. What will be the balance of the account in 10 years? Assume that he makes no additional deposits and no withdrawals.

2. INTEREST Janie’s bank pays 2.8% annual interest compounded continuously on savings accounts. She placed $2000 in the account. How long will it take for her initial deposit to double in value? Assume that she makes no additional deposits and no withdrawals. Round your answer to the nearest quarter year.

3. POPULATION The equation \( A = A_0e^{rt} \) describes the growth of the world’s population where \( A \) is the population at time \( t \), \( A_0 \) is the population at \( t = 0 \), and \( r \) is the annual growth rate. The world’s population at the start of 2008 was estimated at 6,641,000,000. If the annual growth rate is 1.2%, when will the world population reach 9 billion?

4. BACTERIA A bacterial population grows exponentially, doubling every 72 hours.

<table>
<thead>
<tr>
<th>bacteria</th>
<th>x</th>
<th>2x</th>
<th>4x</th>
<th>8x</th>
</tr>
</thead>
<tbody>
<tr>
<td>time</td>
<td>0</td>
<td>72</td>
<td>144</td>
<td>216</td>
</tr>
</tbody>
</table>

Let \( P \) be the initial population size and let \( t \) be time in hours. Use the equation from Exercise 3 to write a formula using the natural base exponential function that gives the size of the population \( y \) as a function of \( P \) and \( t \).

5. MONEY MANAGEMENT Linda wants to invest $20,000. She is looking at two possible accounts. Account A is a standard savings account that pays 3.4% annual interest compounded continuously. Account B would pay her a fixed amount of $200 every quarter.

a. If Linda can invest the money for 5 years only, which account would give her the higher return on her investment? How much more money would she make by choosing the higher paying account?

b. If Linda can invest the money for 10 years only, which account would give her the higher return on her investment? How much more money would she make by choosing the higher paying account?

c. If Linda can invest the money for 20 years only, which account would give her the higher return on her investment? How much more money would she make by choosing the higher paying account?
Exponential Growth and Decay

| Exponential Growth | \( f(x) = ae^{kt} \) where \( a \) is the initial value of \( y \), \( t \) is time in years, and \( k \) is a constant representing the rate of continuous growth. |
| Exponential Decay | \( f(x) = ae^{-kt} \) where \( a \) is the initial value of \( y \), \( t \) is time in years, and \( k \) is a constant representing the rate of continuous decay. |

**Example**

**POPULATION** In 2000, the world population was estimated to be 6.124 billion people. In 2005, it was 6.515 billion.

a. Determine the value of \( k \), the world’s relative rate of growth.

\[
y = ae^{kt} \quad \text{Formula for continuous growth}
\]

\[
6.515 = 6.124e^{k(5)}
\]

\[
y = 6.515, \ a = 6.124, \ \text{and} \ t = 2005 - 2000 = 5
\]

\[
\frac{6.515}{6.124} = e^{5k}
\]

Divide each side by 6.124.

\[
\ln \frac{6.515}{6.124} = \ln e^{5k}
\]

Property of Equality for Logarithmic Functions

\[
\ln \frac{6.515}{6.124} = 5k
\]

\[
\ln \frac{6.515}{6.124} = 5k
\]

Divide each side by 5 and use a calculator.

\[
0.01238 \approx k
\]

The world’s relative rate of growth is about 0.01238 or 1.2%

b. When will the world’s population reach 7.5 billion people?

\[
7.5 \approx 6.124e^{0.01238t}
\]

\[
y = 7.5, \ a = 6.124, \ \text{and} \ k = 0.01238
\]

\[
\frac{7.5}{6.124} \approx e^{0.01238t}
\]

Divide each side by 6.124.

\[
\ln \frac{7.5}{6.124} \approx \ln e^{0.01238t}
\]

Property of Equality for Logarithmic Functions

\[
\ln \frac{7.5}{6.124} \approx 0.01238t
\]

\[
\ln \frac{7.5}{6.124} \approx 0.01238t
\]

Divide each side by 0.01238 and use a calculator.

\[
16.3722 \approx t
\]

The world’s population will reach 7.5 billion in 2016.

**Exercise**

1. **CARBON DATING** Use the formula \( y = ae^{-0.00012t} \), where \( a \) is the initial amount of carbon 14, \( t \) is the number of years ago the animal lived, and \( y \) is the remaining amount after \( t \) years.

   a. How old is a fossil that has lost 95% of its Carbon-14?

   b. How old is a skeleton that has 95% of its Carbon-14 remaining?
Logistic Growth  A logistic function models the S-curve of growth of some set \( \lambda \). The initial stage of growth is approximately exponential; then, as saturation begins, the growth slows, and at some point, growth stops.

**Example**  The population of a certain species of fish in a lake after \( t \) years is given by \( P(t) = \frac{1880}{1 + 1.42e^{-0.037t}} \).

a. Graph the function.

![Graph of the function](image)

b. Find the horizontal asymptote. What does it represent in the situation?  
   The horizontal asymptote is \( P(t) = 1880 \). The population of fish will reach a ceiling of 1880.

c. When will the population reach 1875?

   \[
   1875 = \frac{1880}{1 + 1.42e^{-0.037t}}
   \]

   \[
   1875(1 + 1.42e^{-0.037t}) = 1880
   \]

   \[
   2662.5e^{-0.037t} = 5
   \]

   \[
   e^{-0.037t} = \frac{5}{2662.5}
   \]

   \[
   -0.037t = \ln \left( \frac{5}{2662.5} \right)
   \]

   \[
   t = \frac{\ln \left( \frac{5}{2662.5} \right)}{-0.037}
   \]

   \[
   t \approx 169.66
   \]

   The population will reach 1875 in about 170 years.

**Exercise**

1. Assume the population of gnats in a specific habitat follows the function \( P(t) = \frac{17,000}{1 + 15e^{-0.0082t}} \).

   a. Graph the function for \( t \geq 0 \).

   ![Graph of the function](image)

   b. What is the horizontal asymptote?

   c. What is the maximum population?

   d. When does the population reach 15,000?
1. BACTERIA  How many hours will it take a culture of bacteria to increase from 20 to 2000? Use \( k = 0.614 \).

2. RADIOACTIVE DECAY  A radioactive substance has a half-life of 32 years. Find the constant \( k \) in the decay formula for the substance.

3. RADIOACTIVE DECAY  Cobalt, an element used to make alloys, has several isotopes. One of these, cobalt 60, is radioactive and has a half-life of 5.7 years. Cobalt 60 is used to trace the path of nonradioactive substances in a system. What is the value of \( k \) for cobalt 60?

4. WHALES  Modern whales appeared 5–10 million years ago. The vertebrae of a whale discovered by paleontologists contain roughly 0.25% as much carbon-14 as they would have contained when the whale was alive. How long ago did the whale die? Use \( k = 0.00012 \).

5. POPULATION  The population of rabbits in an area is modeled by the growth equation \( P(t) = 8e^{0.26t} \), where \( P \) is in thousands and \( t \) is in years. How long will it take for the population to reach 25,000?

6. RADIOACTIVE DECAY  A radioactive element decays exponentially. The decay model is given by the formula \( A = A_0e^{-0.04463t} \). \( A \) is the amount present after \( t \) days and \( A_0 \) is the amount present initially. Assume you are starting with 50g. How much of the element remains after 10 days? 30 days?

7. POPULATION  A population is growing continuously at a rate of 3\%. If the population is now 5 million, what will it be in 17 years’ time?

8. BACTERIA  A certain bacteria is growing exponentially according to the model \( y = 80e^{kt} \). Using \( k = 0.071 \), find how many hours it will take for the bacteria reach a population of 10,000 cells?

9. LOGISTIC GROWTH  The population of a certain habitat follows the function

\[
P(t) = \frac{16,300}{(1 + 17.5e^{-0.065t})}.
\]

a. What is the maximum population?

b. When does the population reach 16,200?
1. **PROGRAMMING** For reasons having to do with speed, a computer programmer wishes to model population size using a natural base exponential function. However, the programmer is told that the users of the program will be thinking in terms of the annual percentage increase. Let \( r \) be the percentage that the population increases each year. Find the value for \( k \) in terms of \( r \) so that \( e^k = 1 + r \).

2. **CARBON DATING** Archeologists uncover an ancient wooden tool. They analyze the tool and find that it has 22% as much carbon 14 compared to the likely amount that it contained when it was made. Given that the half-life of carbon 14 is about 5730 years, about how old is the artifact? Round your answer to the nearest 100 years.

3. **POPULATION** The doubling time of a population is \( d \) years. The population size \( y \) can be modeled by an exponential equation of the form \( y = ae^{kt} \), where \( a \) is the initial population size and \( t \) is time. What is \( k \) in terms of \( d \)?

4. **POPULATION** Louisa read that the population of her town has increased steadily each year. Today, the population of her town has grown to 68,735. One year ago, the population was 67,387. Based on this information, what was the population of her town 100 years ago?

5. **CONSUMER AWARENESS** Jason wants to buy a brand new high-definition (HD) television. He could buy one now because he has $7000 to spend, but he thinks that if he waits, the quality of HD televisions will improve. His $7000 earns 2.5% interest annually compounded continuously. The television he wants to buy costs $5000 now, but the cost increases each year by 7%.

   a. Write a natural base exponential function that gives the value of Jason’s account as a function of time \( t \).

   b. Write a natural base exponential function that gives the cost of the television Jason wants as a function of time \( t \).

   c. In how many years will the cost of the television exceed the value of the money in Jason’s account? In other words, how much time does Jason have to decide whether he wants to buy the television? Round your answer to the nearest tenth of a year.

6. **LOGISTIC GROWTH** The population of a bacteria can be modeled by \( P(t) = \frac{22,000}{1 + 1.2e^{-kt}} \) where \( t \) is time in hours and \( k \) is a constant.

   a. After 1 hour the bacteria population is 10,532, what is the value of \( k \)?

   b. When does the population reach 21,900?
### 9-1 Study Guide

**Multiplying and Dividing Rational Expressions**

**Simplify Rational Expressions** A ratio of two polynomial expressions is a **rational expression**. To simplify a rational expression, divide both the numerator and the denominator by their greatest common factor (GCF).

<table>
<thead>
<tr>
<th>Multiplying Rational Expressions</th>
<th>For all rational expressions ( \frac{a}{b} ) and ( \frac{c}{d} ), ( \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} ), if ( b \neq 0 ) and ( d \neq 0 ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dividing Rational Expressions</td>
<td>For all rational expressions ( \frac{a}{b} ) and ( \frac{c}{d} ), ( \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} ), if ( b \neq 0 ), ( c \neq 0 ), and ( d \neq 0 ).</td>
</tr>
</tbody>
</table>

#### Example

**Simplify each expression.**

a. \( \frac{24a^5b^2}{(2ab)^4} \)

\[
\frac{24a^5b^2}{(2ab)^4} = \frac{24a^5b^2}{16a^4b^4} = \frac{3a}{b^2}
\]

b. \( \frac{3r^2n^3}{5t^4} \cdot \frac{20t^2}{9r^3n} \)

\[
\frac{3r^2n^3}{5t^4} \cdot \frac{20t^2}{9r^3n} = \frac{2 \cdot 2 \cdot n \cdot n}{3 \cdot r \cdot r} = \frac{4n^2}{3r^2}
\]

c. \( \frac{x^2 + 8x + 16}{2x - 2} \div \frac{x^2 + 2x - 8}{x - 1} \)

\[
\frac{x^2 + 8x + 16}{2x - 2} \div \frac{x^2 + 2x - 8}{x - 1} = \frac{x^2 + 8x + 16}{2x - 2} \cdot \frac{x - 1}{x^2 + 2x - 8} = \frac{(x + 4)(x + 4)}{2(x - 1)} = \frac{x + 4}{2(x - 1)}
\]

#### Exercises

**Simplify each expression.**

1. \( \frac{(-2ab)^3}{20ab^4} \)
2. \( \frac{4x^2 - 12x + 9}{9 - 6x} \)
3. \( \frac{x^2 + x - 6}{x^2 - 6x - 27} \)

4. \( \frac{3m^3 - 3m}{6m^4} \cdot \frac{4m^5}{m + 1} \)
5. \( \frac{c^2 - 3c}{c^2 - 25} \cdot \frac{c^2 + 4c - 5}{c^2 - 4c + 3} \)

6. \( \frac{(m - 3)^2}{m^2 - 6m + 9} \cdot \frac{m^3 - 9m}{m^2 - 9} \)
7. \( \frac{6xy^4}{25x^3} \div \frac{18x^2}{5y} \)

8. \( \frac{16p^2 - 8p + 1}{14p^3} \div \frac{4p^2 + 7p - 2}{7p^3} \)
9. \( \frac{2m - 1}{m^2 - 3m - 10} \div \frac{4m^2 - 1}{4m + 8} \)
Multiplying and Dividing Rational Expressions

Simplify Complex Fractions A complex fraction is a rational expression with a numerator and/or denominator that is also a rational expression. To simplify a complex fraction, first rewrite it as a division problem.

Example

Simplify \( \frac{\frac{3n - 1}{n}}{\frac{3n^2 + 8n - 3}{n^4}} \).

\[
\begin{align*}
\frac{\frac{3n - 1}{n}}{\frac{3n^2 + 8n - 3}{n^4}} & = \frac{3n - 1}{n} \div \frac{3n^2 + 8n - 3}{n^4} \\
& = \frac{3n - 1}{n} \cdot \frac{n^4}{3n^2 + 8n - 3} \\
& = \frac{1}{n(3n - 1)(n + 3)} \\
& = \frac{n^3}{n + 3}
\end{align*}
\]

Exercises

Simplify each expression.

1. \( \frac{x^3y^2}{a^2b^2} \div \frac{ax^2y}{b^2} \)

2. \( \frac{a^2bc^3}{ab^2} \div \frac{x^3y^2}{c^4x^2y} \)

3. \( \frac{b^2 - 1}{3b + 2} \div \frac{b + 1}{3b^2 - b - 2} \)

4. \( \frac{b^2 - 100}{b^2} \div \frac{3b^2 - 31b + 10}{2b} \)

5. \( \frac{x - 4}{x^2 + 6x + 9} \div \frac{x^2 - 2x - 8}{3 + x} \)

6. \( \frac{a^2 - 16}{a + 2} \div \frac{a^2 + 3a - 4}{a^2 + a - 2} \)

7. \( \frac{2x^2 + 9x + 9}{x + 1} \div \frac{10x^2 + 19x + 6}{5x^2 + 7x + 2} \)

8. \( \frac{b + 2}{b^2 - 6b + 8} \div \frac{b^2 + b - 2}{b^2 - 16} \)

9. \( \frac{x^2 - x - 2}{x^2 + x - 6} \div \frac{x + 1}{x + 3} \)
9-1 Practice

Multiplying and Dividing Rational Expressions

Simplify each expression.

1. \( \frac{9a^2b^3}{27a^4b^4c} \)
2. \( \frac{(2m^3n^3)^3}{-18m^5n^4} \)
3. \( \frac{10y^2 + 15y}{35y^2 - 5y} \)

4. \( \frac{2k^2 - k - 15}{k^2 - 9} \)
5. \( \frac{25 - v^2}{3v^2 - 13v - 10} \)

6. \( \frac{x^4 + x^3 - 2x^2}{x^4 - x^3} \)
7. \( -\frac{2u^3y}{15xz^5} \cdot \frac{25x^5}{14u^2y^2} \)

8. \( \frac{a + y}{6} \cdot \frac{4}{y + a} \)
9. \( \frac{n^5}{n - 6} \cdot \frac{n^2 - 6n}{n^8} \)

10. \( \frac{a - y}{w + n} \cdot \frac{w^2 - n^2}{y - a} \)
11. \( \frac{x^2 - 5x - 24}{6x + 2x^2} \cdot \frac{5x^2}{8 - x} \)

12. \( \frac{x - 5}{10x - 2} \cdot \frac{25x^2 - 1}{x^2 - 10x + 25} \)
13. \( \frac{a^3y^3}{w^7} \div \frac{a^3w^2}{w^5y^2} \)

14. \( \left( \frac{2xy}{w^2} \right)^3 \div \frac{24x^2}{w^5} \)
15. \( \frac{x + y}{6} \div \frac{x^2 - y^2}{3} \)

16. \( \frac{3x + 6}{x^2 - 9} \div \frac{6x^2 + 12x}{4x + 12} \)
17. \( \frac{2s^2 - 7s - 15}{(s + 4)^2} \div \frac{s^2 - 10s + 25}{s + 4} \)

18. \( \frac{9 - a^2}{a^2 + 5a + 6} \div \frac{2a - 6}{5a + 10} \)
19. \( \frac{2x + 1}{\frac{x}{4 - x}} \)

20. \( \frac{x^2 - 9}{3 - x} \div \frac{4}{8} \)
21. \( \frac{x^3 + 2^3}{x^2 - 2x} \div \frac{(x + 2)^3}{x^2 + 4x + 4} \)

22. GEOMETRY A right triangle with an area of \( x^2 - 4 \) square units has a leg that measures \( 2x + 4 \) units. Determine the length of the other leg of the triangle.

23. GEOMETRY A rectangular pyramid has a base area of \( \frac{x^2 + 3x - 10}{2x} \) square centimeters and a height of \( \frac{x^2 - 3x}{x^2 - 5x + 6} \) centimeters. Write a rational expression to describe the volume of the rectangular pyramid.
1. **JELLY BEANS** A large vat contains \( G \) green jelly beans and \( R \) red jelly beans. A bag of 100 red and 100 green jelly beans is added to the vat. What is the new ratio of red to green jelly beans in the vat?

2. **MILEAGE** Beth drives a hybrid car that gets 45 miles per gallon in the city and 48 miles per gallon on the highway. Beth uses \( C \) gallons of gas in the city and \( H \) gallons of gas on the highway. Write an expression for the average number of miles per gallon that Beth gets with her car in terms of \( C \) and \( H \).

3. **HEIGHT** The front face of a Nordic house is triangular. The surface area of the face is \( x^2 + 3x + 10 \) where \( x \) is the base of the triangle.

What is the height of the triangle in terms of \( x \)?

4. **OIL SLICKS** David was moving a drum of oil around his circular outdoor pool when the drum cracked, and oil spilled into the pool. The oil spread itself evenly over the surface of the pool. Let \( V \) denote the volume of oil spilled and let \( r \) be the radius of the pool. Write an equation for the thickness of the oil layer.

5. **RUNNING** Harold runs to the local food mart to buy a gallon of soy milk. Because he is weighed down on his return trip, he runs slower on the way back. He travels \( S_1 \) feet per second on the way to the food mart and \( S_2 \) feet per second on the way back. Let \( d \) be the distance he has to run to get to the food mart. Remember: distance = rate \( \times \) time.

a. Write an equation that gives the total time Harold spent running for this errand.

b. What speed would Harold have to run if he wanted to maintain a constant speed for the entire trip yet take the same amount of time running?
**Adding and Subtracting Rational Expressions**

**LCM of Polynomials** To find the least common multiple of two or more polynomials, factor each expression. The LCM contains each factor the greatest number of times it appears as a factor.

**Example 1** Find the LCM of $16pq^4r$, $40pq^4r^2$, and $15p^3r^4$.

1. $16pq^4r = 2^4 \cdot p^1 \cdot q^4 \cdot r$  
2. $40pq^4r^2 = 2^3 \cdot 5^1 \cdot p^1 \cdot q^4 \cdot r^2$  
3. $15p^3r^4 = 3^1 \cdot 5^1 \cdot p^3 \cdot r^4$  

LCM = $2^4 \cdot 3^1 \cdot 5^1 \cdot p^3 \cdot q^4 \cdot r^4$  

= $240p^3q^4r^4$

**Example 2** Find the LCM of $3m^2 - 3m - 6$ and $4m^2 + 12m - 40$.

1. $3m^2 - 3m - 6 = 3(m + 1)(m - 2)$  
2. $4m^2 + 12m - 40 = 4(m - 2)(m + 5)$

LCM = $12(m + 1)(m - 2)(m + 5)$

**Exercises**

Find the LCM of each set of polynomials.

1. $14ab^2$, $42bc^3$, $18a^2c$  
2. $8cdf^3$, $28e^2f$, $35d^4f^2$

3. $65xy^2$, $10x^2y^2$, $26y^4$  
4. $11mn^5$, $18m^2n^3$, $20mn^4$

5. $15a^4b$, $50a^2b^2$, $40b^8$  
6. $24pq^7$, $30p^2q^2$, $45pq^3$

7. $39b^2c^2$, $52b^4c$, $12c^3$  
8. $12xy^4$, $42x^2y$, $30x^2y^3$

9. $56stv^2$, $24s^2v^2$, $70t^2v^3$  
10. $x^2 + 3x$, $10x^2 + 25x - 15$

11. $9x^2 - 12x + 4$, $3x^2 + 10x - 8$  
12. $22x^2 + 66x - 220$, $4x^2 - 16$

13. $8x^2 - 36x - 20$, $2x^2 + 2x - 60$  
14. $5x^2 - 125$, $5x^2 + 24x - 5$

15. $3x^2 - 18x + 27$, $2x^3 - 4x^2 - 6x$  
16. $45x^2 - 6x - 3$, $45x^2 - 5$

17. $x^3 + 4x^2 - x - 4$, $x^2 + 2x - 3$  
18. $54x^3 - 24x$, $12x^2 - 26x + 12$
Adding and Subtracting Rational Expressions

**Add and Subtract Rational Expressions** To add or subtract rational expressions, follow these steps.

1. **Step 1** Find the least common denominator (LCD). Rewrite each expression with the LCD.
2. **Step 2** Add or subtract the numerators.
3. **Step 3** Combine any like terms in the numerator.
4. **Step 4** Factor if possible.
5. **Step 5** Simplify if possible.

**Example** Simplify \( \frac{6}{2x^2 + 2x - 12} + \frac{2}{x^2 - 4} \).

\[
\begin{align*}
&= \frac{6}{2(x + 3)(x - 2)} + \frac{2}{(x - 2)(x + 2)} \\
&= \frac{6(x + 2)}{2(x + 3)(x - 2)(x + 2)} + \frac{2 \cdot 2(x + 3)}{2(x + 3)(x - 2)(x + 2)} \\
&= \frac{6(x + 2) - 4(x + 3)}{2(x + 3)(x - 2)(x + 2)} \\
&= \frac{6x + 12 - 4x - 12}{2(x + 3)(x - 2)(x + 2)} \\
&= \frac{2x}{2(x + 3)(x - 2)(x + 2)} \\
&= \frac{x}{(x + 3)(x - 2)(x + 2)}
\end{align*}
\]

**Exercises**

Simplify each expression.

1. \( \frac{-7xy}{3x} + \frac{4y^2}{2y} \)
2. \( \frac{2}{x - 3} - \frac{1}{x - 1} \)
3. \( \frac{4a}{3bc} - \frac{15b}{5ac} \)
4. \( \frac{3}{x + 2} + \frac{4x + 5}{3x + 6} \)
5. \( \frac{3x + 3}{x^2 + 2x + 1} + \frac{x - 1}{x^2 - 1} \)
6. \( \frac{4}{4x^2 - 4x + 1} - \frac{5x}{20x^2 - 5} \)
9-2 Practice

Adding and Subtracting Rational Expressions

Find the LCM of each set of polynomials.

1. \(x^2y, xy^3\)  
2. \(a^2b^3c, abc^4\)  
3. \(x + 1, x + 3\)  
4. \(g - 1, g^2 + 3g - 4\)  
5. \(2r + 2, r^2 + r, r + 1\)  
6. \(3, 4w + 2, 4w^2 - 1\)  
7. \(x^2 + 2x - 8, x + 4\)  
8. \(x^2 - x - 6, x^2 + 6x + 8\)  
9. \(d^2 + 6d + 9, 2(d^2 - 9)\)

Simplify each expression.

10. \(\frac{5}{6ab} - \frac{7}{8a}\)  
11. \(\frac{5}{12xy} - \frac{1}{5x^2y^2}\)  
12. \(\frac{1}{6c^2}d + \frac{3}{4cd^3}\)  
13. \(\frac{4m}{3mn} + 2\)  
14. \(2x - 5 - \frac{x - 8}{x + 4}\)  
15. \(\frac{4}{a - 3} + \frac{9}{a - 5}\)  
16. \(\frac{16}{x^2 - 16} + \frac{2}{x + 4}\)  
17. \(\frac{2 - 5m}{m - 9} + \frac{4m - 5}{9 - m}\)  
18. \(\frac{y - 5}{y^2 - 3y - 10} + \frac{y}{y^2 + y - 2}\)  
19. \(\frac{5}{2x - 12} - \frac{20}{x^2 - 4x - 12}\)  
20. \(\frac{2p - 3}{p^2 - 5p + 6} - \frac{5}{p^2 - 9}\)  
21. \(\frac{1}{5n} - \frac{3}{4} + \frac{7}{10n}\)  
22. \(\frac{2a}{a - 3} - \frac{2a}{a + 3} + \frac{36}{a^2 - 9}\)  
23. \(\frac{2}{x - y} + \frac{1}{x + y}\)  
24. \(\frac{r + 6}{r^2 + 4r + 3} - \frac{1}{r^2 + 2r}\)

25. GEOMETRY The expressions \(\frac{5x}{2}, \frac{20}{x + 4}, \text{ and } \frac{10}{x - 4}\) represent the lengths of the sides of a triangle. Write a simplified expression for the perimeter of the triangle.

26. KAYAKING Mai is kayaking on a river that has a current of 2 miles per hour. If \(r\) represents her rate in calm water, then \(r + 2\) represents her rate with the current, and \(r - 2\) represents her rate against the current. Mai kayaks 2 miles downstream and then back to her starting point. Use the formula for time, \(t = \frac{d}{r}\), where \(d\) is the distance, to write a simplified expression for the total time it takes Mai to complete the trip.
9-2 Word Problem Practice

Adding and Subtracting Rational Expressions

1. SQUARES Susan’s favorite perfect square is \( s^2 \) and Travis’ is \( t^2 \), where \( s \) and \( t \) are whole numbers. What perfect square is guaranteed to be divisible by both Susan’s and Travis’ favorite perfect squares regardless of their specific value?

2. ELECTRIC POTENTIAL The electrical potential function between two electrons is given by a formula that has the form \( \frac{1}{r} + \frac{1}{1-r} \). Simplify this expression.

3. TRAPEZOIDS The cross section of a stand consists of two trapezoids stacked one on top of the other.

![Diagram of a cross section with trapezoids]

The total area of the cross section is \( x^2 \) square units. Assuming the trapezoids have the same height, write an expression for the height of the stand in terms of \( x \). Put your answer in simplest form. (Recall that the area of a trapezoid with height \( h \) and bases \( b_1 \) and \( b_2 \) is given by \( \frac{1}{2} h(b_1 + b_2) \).)

4. FRACTIONS In the seventeenth century, Lord Brouncker wrote down a most peculiar mathematical equation:

\[
\frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \frac{9^2}{2 + \ldots}}}}}.
\]

This is an example of a continued fraction. Simplify the continued fraction

\[
n + \frac{1}{n + \frac{1}{n}}.
\]

5. RELAY RACE Mark, Connell, Zack, and Moses run the 400 meter relay together. Each of them runs 100 meters. Their average speeds were \( s \), \( s + 0.5 \), \( s - 0.5 \), and \( s - 1 \) meters per second, respectively.

a. What were their individual times for their own legs of the race?

b. Write an expression for their time as a team. Write your answer as a ratio of two polynomials.

c. The world record for the 100 meter relay is 37.4 seconds. What will \( s \) equal if the team ties the world record?
Graphing Reciprocal Functions

Vertical and Horizontal Asymptotes

<table>
<thead>
<tr>
<th>Parent Function of Reciprocal Functions</th>
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</thead>
<tbody>
<tr>
<td>Parent Function</td>
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<tr>
<td>Type of Graph</td>
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<tr>
<td>Domain</td>
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<td>Range</td>
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<td>Symmetry</td>
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<tr>
<td>Intercepts</td>
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<tr>
<td>Asymptotes</td>
</tr>
</tbody>
</table>

**Example**  
Identify the asymptotes, domain, and range of the function  
\[ f(x) = \frac{3}{x+2} \].

Identify \( x \) values for which \( f(x) \) is undefined.  
\( x + 2 = 0 \), so \( x = -2 \). \( f(x) \) is not defined when \( x = -2 \), so there is an asymptote at \( x = -2 \).  
From \( x = -2 \), as \( x \)-values decrease, \( f(x) \) approaches 0.  
As \( x \)-values increase, \( f(x) \) approaches 0. So there is an asymptote at \( f(x) = 0 \).  
The domain is all real numbers not equal to \(-2\), and the range is all real numbers not equal to 0.

**Exercises**

Identify the asymptotes, domain, and range of each function.

1. \( f(x) = \frac{1}{x} \)  
2. \( f(x) = \frac{-3}{x-1} \)  
3. \( f(x) = \frac{4}{x+1} + 2 \)
Graphing Reciprocal Functions

Transformations of Reciprocal Functions

<table>
<thead>
<tr>
<th>Equation Form</th>
<th>( f(x) = \frac{a}{x - h} + k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal Translation</td>
<td>The <em>vertical</em> asymptote moves to ( x = h ).</td>
</tr>
<tr>
<td>Vertical Translation</td>
<td>The <em>horizontal</em> asymptote moves to ( y = k ).</td>
</tr>
<tr>
<td>Reflection</td>
<td>The graph is reflected across the ( x )-axis when ( a &lt; 0 ).</td>
</tr>
<tr>
<td>Compression and Expansion</td>
<td>The graph is compressed vertically when (</td>
</tr>
</tbody>
</table>

**Example**

Graph \( f(x) = \frac{-1}{x + 1} - 3 \). State the domain and range.

- \( a < 0 \): The graph is reflected over the \( x \)-axis.
- \( 0 < |a| < 1 \): The graph is compressed vertically.
- \( h = -1 \): The *vertical* asymptote is at \( x = -1 \).
- \( k = -3 \): The *horizontal* asymptote is at \( f(x) = -3 \).
- \( D = \{x \mid x \neq -1\}; \ R = \{f(x) \mid f(x) \neq -3\} \)

**Exercises**

Graph each function. State the domain and range.

1. \( f(x) = \frac{1}{x + 1} \)
2. \( f(x) = \frac{-2}{x - 2} \)
3. \( f(x) = \frac{-1}{x - 3} \)
4. \( f(x) = \frac{1}{x + 5} + 3 \)
5. \( f(x) = \frac{-2}{x - 1} + 2 \)
6. \( f(x) = \frac{1}{x - 3} + 4 \)
Graphing Reciprocal Functions

Identify the asymptotes, domain, and range of each function.

1. \( f(x) = \frac{1}{x - 1} - 3 \)
2. \( f(x) = \frac{1}{x + 1} + 3 \)
3. \( f(x) = \frac{-3}{x - 2} + 5 \)

Graph each function. State the domain and range.

4. \( f(x) = \frac{1}{x + 1} - 5 \)
5. \( f(x) = \frac{-1}{x - 3} - 4 \)
6. \( f(x) = \frac{3}{x - 2} + 4 \)

7. RACE Kate enters a 120-mile bicycle race. Her basic rate is 10 miles per hour, but Kate will average \( x \) miles per hour faster than that. Write and graph an equation relating \( x \) (Kate’s speed beyond 10 miles per hour) to the time it would take to complete the race. If she wanted to finish the race in 4 hours instead of 5 hours, how much faster should she travel?
1. VACATION The Porter family takes a trip and rents a car. The rental costs $125 plus $0.30 per mile.

   a. Write the equation that relates the cost per mile to the number of miles traveled.

   b. Explain any limitations to the range or domain in this situation.

2. PLANES A plane is scheduled to leave Dallas for an 800-mile flight to Chicago’s O’Hare airport at time $t = 0$. The departure is delayed for two hours. Write two equations that represent the planes’ speed, $r$, on the vertical axis as a function of travel time, $t$, on the horizontal axis. Graph the equations below. How do the two curves relate?

3. BIOLOGY A rabbit population follows the function $P(t) = \frac{40}{t + 2} + 10$, with $P(t)$ equal to the rabbit population after $t$ months. Eventually, what happens to the rabbit population?

4. COMPUTERS To make computers, a company must pay $5000 for rent and overhead and $435 per computer for parts.

   a. Write the equation relating average cost to make a computer to how many computers are being made.

   b. Graph the function you found in part a.

   c. What is the minimum number of computers the company needs to make so that the average cost is less than $685?
Graphing Rational Functions

Vertical and Horizontal Asymptotes

<table>
<thead>
<tr>
<th>Rational Function</th>
<th>A function with an equation of the form ( f(x) = \frac{p(x)}{q(x)} ), where ( p(x) ) and ( q(x) ) are polynomial expressions and ( q(x) \neq 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain</td>
<td>The domain of a rational function is limited to values for which the function is defined.</td>
</tr>
<tr>
<td>Vertical Asymptote</td>
<td>An asymptote is a line that the graph of a function approaches. If the simplified form of the related rational expression is undefined for ( x = a ), then ( x = a ) is a vertical asymptote.</td>
</tr>
<tr>
<td>Horizontal Asymptote</td>
<td>Often a horizontal asymptote occurs in the graph of a rational function where a value is excluded from the range.</td>
</tr>
</tbody>
</table>

**Example**

Graph \( f(x) = \frac{x^2 + x - 6}{x + 1} \).

\[
\frac{x^2 + x - 6}{x + 1} = \frac{(x + 3)(x - 2)}{x + 1}
\]

Therefore the graph of \( f(x) \) has zeroes at \( x = -3 \) and \( x = 2 \) and a vertical asymptote at \( x = -1 \). Because the degree of \( x^2 + x - 6 \) is greater than \( x + 1 \), there is no horizontal asymptote. Make a table of values. Plot the points and draw the graph.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-3.5</td>
<td>-2</td>
<td>0</td>
<td>4</td>
<td>-6</td>
<td>-2</td>
<td>0</td>
<td>1.5</td>
<td>2.8</td>
</tr>
</tbody>
</table>

**Exercises**

Graph each function.

1. \( f(x) = \frac{4}{x^2 + 3x - 10} \)

2. \( f(x) = \frac{x^2 - 2x + 1}{x^2 + 2x + 1} \)

3. \( f(x) = \frac{2x + 9}{2x^2 - x - 3} \)
Oblique Asymptotes and Point Discontinuity  
An oblique asymptote is an asymptote that is neither horizontal nor vertical. In some cases, graphs of rational functions may have point discontinuity, which looks like a hole in the graph. That is because the function is undefined at that point.

### Oblique Asymptotes

If \( f(x) = \frac{a(x)}{b(x)} \), \( a(x) \) and \( b(x) \) are polynomial functions with no common factors other than 1 and \( b(x) \neq 0 \), then \( f(x) \) has an oblique asymptote if the degree of \( a(x) \) minus the degree of \( b(x) \) equals 1.

### Point Discontinuity

If \( f(x) = \frac{a(x)}{b(x)} \), \( b(x) \neq 0 \), and \( x - c \) is a factor of both \( a(x) \) and \( b(x) \), then there is a point discontinuity at \( x = c \).

#### Example

Graph \( f(x) = \frac{x - 1}{x^2 + 2x - 3} \).

\[
\frac{x - 1}{x^2 + 2x - 3} = \frac{x - 1}{(x - 1)(x + 3)} \quad \text{or} \quad \frac{1}{x + 3}
\]

Therefore the graph of \( f(x) \) has an asymptote at \( x = -3 \) and a point discontinuity at \( x = 1 \).

Make a table of values. Plot the points and draw the graph.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2.5</th>
<th>-2</th>
<th>-1</th>
<th>-3.5</th>
<th>-4</th>
<th>-5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>2.6</td>
<td>1</td>
<td>0.5</td>
<td>-2</td>
<td>-1</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

### Exercises

Graph each function.

1. \( f(x) = \frac{x^2 + 5x + 4}{x + 3} \)
2. \( f(x) = \frac{x^2 - x - 6}{x - 3} \)
3. \( f(x) = \frac{x^2 - 6x + 8}{x^2 - x - 2} \)
Graph each function.

1. \( f(x) = \frac{-4}{x - 2} \)

2. \( f(x) = \frac{x - 3}{x - 2} \)

3. \( f(x) = \frac{3x}{(x + 3)^2} \)

4. \( f(x) = \frac{2x^2 + 5}{6x - 4} \)

5. \( f(x) = \frac{x^2 + 2x - 8}{x - 2} \)

6. \( f(x) = \frac{x^2 - 7x + 12}{x - 3} \)

7. **PAINTING** Working alone, Tawa can give the shed a coat of paint in 6 hours. It takes her father \( x \) hours working alone to give the shed a coat of paint. The equation \( f(x) = \frac{6 + x}{6x} \) describes the portion of the job Tawa and her father working together can complete in 1 hour. Graph \( f(x) = \frac{6 + x}{6x} \) for \( x > 0, f(x) > 0 \). If Tawa’s father can complete the job in 4 hours alone, what portion of the job can they complete together in 1 hour? What domain and range values are meaningful in the context of the problem?

8. **LIGHT** The relationship between the illumination an object receives from a light source of \( I \) foot-candles and the square of the distance \( d \) in feet of the object from the source can be modeled by \( I(d) = \frac{4500}{d^2} \). Graph the function \( I(d) = \frac{4500}{d^2} \) for \( 0 < I \leq 80 \) and \( 0 < d \leq 80 \). What is the illumination in foot-candles that the object receives at a distance of 20 feet from the light source? What domain and range values are meaningful in the context of the problem?
1. **ROAD TRIP** Robert and Sarah start off on a road trip from the same house. During the trip, Robert’s and Sarah’s cars remain separated by a constant distance. The graph shows the ratio of the distance Sarah has traveled to the distance Robert has traveled. The dotted line shows how this graph would be extended to hypothetical negative values of $x$. What does the $x$-coordinate of the vertical asymptote represent?

2. **GRAPHS** Alma graphed the function $f(x) = \frac{x^2 - 4x}{x - 4}$ below.

   There is a problem with her graph. Explain how to correct it.

3. **NEWTON** Sir Isaac Newton studied the rational function
   
   $f(x) = \frac{ax^2 + bx^2 + cx + d}{x}.$

   Assuming that $d \neq 0$, where will there be a vertical asymptote to the graph of this function?

4. **BATTING AVERAGES** Alex Rodriguez had a lifetime batting average of .305 at the beginning of the 2007 season. He had 2067 hits out of 6767 at bats. During the 2007 season, he had 183 hits.

   a. Write an equation describing Rodriguez’s batting average at the end of the 2007 season using $x$ for the number of at bats he had during the season.

   b. Determine where the horizontal and vertical asymptotes for the graph of the equation would be.

   c. What is the meaning of the horizontal asymptote for the graph of this equation?
Solve Rational Equations

A rational equation contains one or more rational expressions. To solve a rational equation, first multiply each side by the least common denominator of all of the denominators. Be sure to exclude any solution that would produce a denominator of zero.

Example

Solve \( \frac{9}{10} + \frac{2}{x + 1} = \frac{2}{5} \). Check your solution.

\[
\frac{9}{10} + \frac{2}{x + 1} = \frac{2}{5} \quad \text{Original equation}
\]

\[
10(x + 1)(\frac{9}{10} + \frac{2}{x + 1}) = 10(x + 1)(\frac{2}{5}) \quad \text{Multiply each side by 10(x + 1)}.
\]

\[
9(x + 1) + 2(10) = 4(x + 1) \quad \text{Multiply.}
\]

\[
9x + 9 + 20 = 4x + 4 \quad \text{Distribute.}
\]

\[
5x = -25 \quad \text{Subtract 4x and 29 from each side.}
\]

\[
x = -5 \quad \text{Divide each side by 5.}
\]

Check

\[
\frac{9}{10} + \frac{2}{x + 1} = \frac{2}{5} \quad \text{Original equation}
\]

\[
\frac{9}{10} + \frac{2}{-5 + 1} = \frac{2}{5}
\]

\[
\frac{18}{20} - \frac{10}{20} = \frac{2}{5}
\]

\[
\frac{2}{5} = \frac{2}{5} \quad \text{Simplify.}
\]

Exercises

Solve each equation. Check your solution.

1. \( \frac{2y}{3} - \frac{y + 3}{6} = 2 \)

2. \( \frac{4t - 3}{5} - \frac{4 - 2t}{3} = 1 \)

3. \( \frac{2x + 1}{3} - \frac{x - 5}{4} = \frac{1}{2} \)

4. \( \frac{3m + 2}{5m} + \frac{2m - 1}{2m} = 4 \)

5. \( \frac{4}{x - 1} = \frac{x + 1}{12} \)

6. \( \frac{x}{x - 2} + \frac{4}{x - 2} = 10 \)

7. NAVIGATION The current in a river is 6 miles per hour. In her motorboat Marissa can travel 12 miles upstream or 16 miles downstream in the same amount of time. What is the speed of her motorboat in still water? Is this a reasonable answer? Explain.

8. WORK Adam, Bethany, and Carlos own a painting company. To paint a particular house alone, Adam estimates that it would take him 4 days, Bethany estimates 5\( \frac{1}{2} \) days, and Carlos 6 days. If these estimates are accurate, how long should it take the three of them to paint the house if they work together? Is this a reasonable answer?
Solving Rational Equations and Inequalities

Solve Rational Inequalities To solve a rational inequality, complete the following steps.

**Step 1** State the excluded values.

**Step 2** Solve the related equation.

**Step 3** Use the values from steps 1 and 2 to divide the number line into regions. Test a value in each region to see which regions satisfy the original inequality.

**Example**

Solve \(\frac{2}{3n} + \frac{4}{5n} \leq \frac{2}{3}\).

**Step 1** The value of 0 is excluded since this value would result in a denominator of 0.

**Step 2** Solve the related equation.

\[
\begin{align*}
\frac{2}{3n} + \frac{4}{5n} &= \frac{2}{3} \\
15n\left(\frac{2}{3n} + \frac{4}{5n}\right) &= 15n\left(\frac{2}{3}\right) \\
10 + 12 &= 10n \\
22 &= 10n \\
2.2 &= n
\end{align*}
\]

**Step 3** Draw a number with vertical lines at the excluded value and the solution to the equation.

Test \(n = -1\).

\[-\frac{2}{3} + \left(-\frac{4}{5}\right) \leq \frac{2}{3} \text{ is true.}\]

Test \(n = 1\).

\[\frac{2}{3} + \frac{4}{5} \leq \frac{2}{3} \text{ is not true.}\]

Test \(n = 3\).

\[\frac{2}{9} + \frac{4}{15} \leq \frac{2}{3} \text{ is true.}\]

The solution is \(n < 0 \text{ or } n \geq 2.2\).

**Exercises**

Solve each inequality. Check your solutions.

1. \(\frac{3}{a + 1} \geq 3\)
2. \(\frac{1}{x} \geq 4x\)
3. \(\frac{1}{2p} + \frac{4}{5p} > \frac{2}{3}\)
4. \(\frac{3}{2x} - \frac{2}{x} > \frac{1}{4}\)
5. \(\frac{4}{x - 1} + \frac{5}{x} < 2\)
6. \(\frac{3}{x^2 - 1} + 1 > \frac{2}{x - 1}\)
Solve each equation or inequality. Check your solutions.

1. \( \frac{12}{x} + \frac{3}{4} = \frac{3}{2} \)
2. \( \frac{x}{x - 1} - 1 = \frac{x}{2} \)
3. \( \frac{p + 10}{p^2 - 2} = \frac{4}{p} \)
4. \( \frac{s}{s + 2} + s = \frac{5s + 8}{s + 2} \)
5. \( \frac{5}{y - 5} = \frac{y}{y - 5} - 1 \)
6. \( \frac{1}{3x - 2} + \frac{5}{x} = 0 \)
7. \( \frac{5}{t} < \frac{9}{2t + 1} \)
8. \( \frac{1}{2h} + \frac{5}{h} = \frac{3}{h - 1} \)
9. \( \frac{4}{w - 2} = \frac{-1}{w + 3} \)
10. \( 5 - \frac{3}{a} < \frac{7}{a} \)
11. \( \frac{4}{5x} + \frac{1}{10} < \frac{3}{2x} \)
12. \( 8 + \frac{3}{y} > \frac{19}{y} \)
13. \( \frac{4}{p} + \frac{1}{3p} < \frac{1}{5} \)
14. \( \frac{6}{x - 1} = \frac{4}{x - 2} + \frac{2}{x + 1} \)
15. \( g + \frac{g}{g - 2} = \frac{2}{g - 2} \)
16. \( b + \frac{2b}{b - 1} = 1 - \frac{b - 3}{b - 1} \)
17. \( \frac{1}{n + 2} + \frac{1}{n - 2} = \frac{3}{n^2 - 4} \)
18. \( \frac{c + 1}{c - 3} = 4 - \frac{12}{c^2 - 2c - 3} \)
19. \( \frac{3}{k - 3} + \frac{4}{k - 4} = \frac{25}{k^2 - 7k + 12} \)
20. \( \frac{4v}{v - 1} - \frac{5v}{v - 2} = \frac{2}{v^2 - 3v + 2} \)
21. \( \frac{y}{y + 2} + \frac{7}{y - 5} = \frac{14}{y^2 - 3y - 10} \)
22. \( \frac{x^2 + 4}{x^2 - 4} + \frac{x}{2 - x} = \frac{2}{x + 2} \)
23. \( \frac{r}{r + 4} + \frac{4}{r - 4} = \frac{r^2 + 16}{r^2 - 16} \)
24. \( 3 = \frac{6a - 1}{2a + 7} + \frac{22}{a + 5} \)

27. **BASKETBALL** Kiana has made 9 of 19 free throws so far this season. Her goal is to make 60% of her free throws. If Kiana makes her next \( x \) free throws in a row, the function \( f(x) = \frac{9 + x}{19 + x} \) represents Kiana’s new ratio of free throws made. How many successful free throws in a row will raise Kiana’s percent made to 60%? Is this a reasonable answer? Explain.

28. **OPTICS** The lens equation \( \frac{1}{p} + \frac{1}{q} = \frac{1}{f} \) relates the distance \( p \) of an object from a lens, the distance \( q \) of the image of the object from the lens, and the focal length \( f \) of the lens. What is the distance of an object from a lens if the image of the object is 5 centimeters from the lens and the focal length of the lens is 4 centimeters? Is this a reasonable answer? Explain.
1. **HEIGHT** Serena can be described as being 8 inches shorter than her sister Malia, or as being 12.5% shorter than Malia. In other words, \( \frac{8}{H + 8} = \frac{1}{8} \), where \( H \) is Serena's height in inches. How tall is Serena?

2. **CRANES** For a wedding, Paula wants to fold 1000 origami cranes.

   She does not want to make anyone fold more than 15 cranes. In other words, if \( N \) is the number of people enlisted to fold cranes, Paula wants \( \frac{1000}{N} \leq 15 \).

   What is the minimum number of people that will satisfy this inequality?

3. **RENTAL** Carlos and his friends rent a car. They split the $200 rental fee evenly. Carlos, together with just two of his friends, decide to rent a portable DVD player as well, and split the $30 rental fee for the DVD player evenly among themselves. Carlos ends up spending $50 for these rentals. Write an equation involving \( N \), the number of friends Carlos has, using this information. Solve the equation for \( N \).

4. **PROJECTILES** A projectile target is launched into the air. A rocket interceptor is fired at the target. The ratio of the altitude of the rocket to the altitude of the projectile \( t \) seconds after the launch of the rocket is given by the formula \( \frac{333t}{-32t^2 + 420t + 27} \). At what time are the target and interceptor at the same altitude?

5. **FLIGHT TIME** The distance between John F. Kennedy International Airport and Los Angeles International Airport is about 2500 miles. Let \( S \) be the airspeed of a jet. The wind speed is 100 miles per hour. Because of the wind, it takes longer to fly one way than the other.

   a. Write an equation for \( S \) if it takes 2 hours and 5 minutes longer to fly between New York and Los Angeles against the wind versus flying with the wind.

   b. Solve the equation you wrote in part a for \( S \).

   c. Write an equation and find how much longer it would take to fly between New York and Los Angeles if the wind speed increases to 150 miles per hour and the airspeed of the jet is 525 miles per hour.
Equations of Circles  The equation of a circle with center \((h, k)\) and radius \(r\) units is \((x - h)^2 + (y - k)^2 = r^2\).

A line is tangent to a circle when it touches the circle at only one point.

Example  Write an equation for a circle if the endpoints of a diameter are at \((-4, 5)\) and \((6, -3)\).

Use the midpoint formula to find the center of the circle.

\[
(h, k) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \quad \text{Midpoint formula}
\]

\[
= \left(\frac{-4 + 6}{2}, \frac{5 + (-3)}{2}\right) \quad (x_1, y_1) = (-4, 5), (x_2, y_2) = (6, -3)
\]

\[
= \left(\frac{2}{2}, \frac{2}{2}\right) \text{ or } (1, 1) \quad \text{Simplify.}
\]

Use the coordinates of the center and one endpoint of the diameter to find the radius.

\[
r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Distance formula}
\]

\[
= \sqrt{(-4 - 1)^2 + (5 - 1)^2} \quad (x_1, y_1) = (1, 1), (x_2, y_2) = (-4, 5)
\]

\[
= \sqrt{(-5)^2 + 4^2} = \sqrt{41} \quad \text{Simplify.}
\]

The radius of the circle is \(\sqrt{41}\), so \(r^2 = 41\).

An equation of the circle is \((x - 1)^2 + (y - 1)^2 = 41\).

Exercises

Write an equation for the circle that satisfies each set of conditions.

1. center \((8, -3)\), radius 6
2. center \((5, -6)\), radius 4
3. center \((-5, 2)\), passes through \((-9, 6)\)
4. center \((3, 6)\), tangent to the \(x\)-axis
5. center \((-4, -7)\), tangent to \(x = 2\)
6. center \((-2, 8)\), tangent to \(y = -4\)
7. center \((7, 7)\), passes through \((12, 9)\)

Write an equation for each circle given the end points of a diameter.

8. \((6, 6)\) and \((10, 12)\)
9. \((-4, -2)\) and \((8, 4)\)
10. \((-4, 3)\) and \((6, -8)\)
Graph Circles  To graph a circle, write the given equation in the standard form of the equation of a circle, \((x - h)^2 + (y - k)^2 = r^2\).

Plot the center \((h, k)\) of the circle. Then use \(r\) to calculate and plot the four points \((h + r, k)\), \((h - r, k)\), \((h, k + r)\), and \((h, k - r)\), which are all points on the circle. Sketch the circle that goes through those four points.

**Example**  Find the center and radius of the circle whose equation is \(x^2 + 2x + y^2 + 4y = 11\). Then graph the circle.

\[
x^2 + 2x + y^2 + 4y = 11
\]
\[
x^2 + 2x + 1 + y^2 + 4y + 4 = 11 + 1 + 4
\]
\[
(x + 1)^2 + (y + 2)^2 = 16
\]

Therefore, the circle has its center at \((-1, -2)\) and a radius of \(\sqrt{16} = 4\). Four points on the circle are \((3, -2), (-5, -2), (-1, 2)\), and \((-1, -6)\).

**Exercises**

Find the center and radius of each circle. Then graph the circle.

1. \((x - 3)^2 + y^2 = 9\)
2. \(x^2 + (y + 5)^2 = 4\)
3. \((x - 1)^2 + (y + 3)^2 = 9\)
4. \((x - 2)^2 + (y + 4)^2 = 16\)
5. \(x^2 + y^2 - 10x + 8y + 16 = 0\)
6. \(x^2 + y^2 - 4x + 6y = 12\)
Write an equation for the circle that satisfies each set of conditions.

1. center \((-4, 2)\), radius 8 units

2. center \((0, 0)\), radius 4 units

3. center \((-\frac{1}{4}, -\sqrt{3})\), radius \(5\sqrt{2}\) units

4. center \((2.5, 4.2)\), radius 0.9 units

5. endpoints of a diameter at \((-2, -9)\) and \((0, -5)\)

6. center at \((-9, -12)\), passes through \((-4, -5)\)

7. center at \((-6, 5)\), tangent to \(x\)-axis

Find the center and radius of each circle. Then graph the circle.

8. \((x + 3)^2 + y^2 = 16\)

9. \(3x^2 + 3y^2 = 12\)

10. \(x^2 + y^2 + 2x + 6y = 26\)

11. \((x - 1)^2 + y^2 + 4y = 12\)

12. \(x^2 - 6x + y^2 = 0\)

13. \(x^2 + y^2 + 2x + 6y = -1\)

14. **WEATHER** On average, the circular eye of a hurricane is about 15 miles in diameter. Gale winds can affect an area up to 300 miles from the storm’s center. A satellite photo of a hurricane’s landfall showed the center of its eye on one coordinate system could be approximated by the point \((80, 26)\).

   a. Write an equation to represent a possible boundary of the hurricane’s eye.

   b. Write an equation to represent a possible boundary of the area affected by gale winds.
1. **RADAR** A scout plane is equipped with radar. The boundary of the radar’s range is given by the circle \((x - 4)^2 + (y - 6)^2 = 4900\). Each unit corresponds to one mile. What is the maximum distance that an object can be from the plane and still be detected by its radar?

2. **STORAGE** An engineer uses a coordinate plane to show the layout of a side view of a storage building. The \(y\)-axis represents a wall and the \(x\)-axis represents the floor. A 10-meter diameter cylinder rests on its side flush against the wall. On the side view, the cylinder is represented by a circle in the first quadrant that is tangent to both axes. Each unit represents 1 meter. What is the equation of this circle?

3. **FERRIS WHEEL** The Texas Star, the largest Ferris wheel in North America, is located in Dallas, Texas. It weighs 678,554 pounds and can hold 264 riders in its 44 gondolas. The Texas Star has a diameter of 212 feet. Use the rectangular coordinate system with the origin on the ground directly below the center of the wheel and write the equation of the circle that models the Texas Star.

4. **POOLS** The pool on an architectural blueprint is given by the equation \(x^2 + 6x + y^2 + 8y = 0\). What point on the edge of the pool is farthest from the origin?

5. **TREASURE** A mathematically inclined pirate decided to hide the location of a treasure by marking it as the center of a circle given by an equation in non-standard form.

   The circle can be represented by:
   \[x^2 + y^2 - 2x + 14y + 49 = 0.\]

   **a.** Rewrite the equation of the circle in standard form.

   **b.** Draw the circle on the map. Where is the treasure?
Solving Linear-Nonlinear Systems

Systems of Equations  Like systems of linear equations, systems of linear-nonlinear equations can be solved by substitution and elimination. If the graphs are a conic section and a line, the system will have 0, 1, or 2 solutions. If the graphs are two conic sections, the system will have 0, 1, 2, 3, or 4 solutions.

Example  Solve the system of equations.  \[ y = x^2 - 2x - 15 \]
\[ x + y = -3 \]

Rewrite the second equation as \[ y = -x - 3 \] and substitute it into the first equation.
\[-x - 3 = x^2 - 2x - 15 \]
\[ 0 = x^2 - x - 12 \]  Add \( x + 3 \) to each side.
\[ 0 = (x - 4)(x + 3) \]  Factor.

Use the Zero Product property to get
\[ x = 4 \quad \text{or} \quad x = -3. \]

Substitute these values for \( x \) in \( x + y = -3 \):
\[ 4 + y = -3 \quad \text{or} \quad -3 + y = -3 \]
\[ y = -7 \quad y = 0 \]

The solutions are \((4, -7)\) and \((-3, 0)\).

Exercises
Solve each system of equations.

1. \[ y = x^2 - 5 \]
\[ y = x - 3 \]

2. \[ x^2 + (y - 5)^2 = 25 \]
\[ y = -x^2 \]

3. \[ x^2 + (y - 5)^2 = 25 \]
\[ y = x^2 \]

4. \[ x^2 + y^2 = 9 \]
\[ x^2 + y = 3 \]

5. \[ x^2 - y^2 = 1 \]
\[ x^2 + y^2 = 16 \]

6. \[ y = x - 3 \]
\[ x = y^2 - 4 \]


**10-7 Study Guide (continued)**

**Solving Linear-Nonlinear Systems**

**Systems of Inequalities** Systems of linear-nonlinear inequalities can be solved by graphing.

**Example 1** Solve the system of inequalities by graphing.

\[ x^2 + y^2 \leq 25 \]
\[ \left( x - \frac{5}{2} \right)^2 + y^2 \geq \frac{25}{4} \]

The graph of \( x^2 + y^2 \leq 25 \) consists of all points on or inside the circle with center \((0, 0)\) and radius 5. The graph of \( \left( x - \frac{5}{2} \right)^2 + y^2 \geq \frac{25}{4} \) consists of all points on or outside the circle with center \( \left( \frac{5}{2}, 0 \right) \) and radius \( \frac{5}{2} \). The solution of the system is the set of points in both regions.

**Example 2** Solve the system of inequalities by graphing.

\[ x^2 + y^2 \leq 25 \]
\[ \frac{y^2}{4} - \frac{x^2}{9} > 1 \]

The graph of \( x^2 + y^2 \leq 25 \) consists of all points on or inside the circle with center \((0, 0)\) and radius 5. The graph of \( \frac{y^2}{4} - \frac{x^2}{9} > 1 \) are the points “inside” but not on the branches of the hyperbola shown. The solution of the system is the set of points in both regions.

**Exercises**

Solve each system of inequalities by graphing.

1. \[ \frac{x^2}{16} + \frac{y^2}{4} \leq 1 \]
   \[ y > \frac{1}{2}x - 2 \]

2. \[ x^2 + y^2 \leq 169 \]
   \[ x^2 + 9y^2 \geq 225 \]

3. \[ y \geq (x - 2)^2 \]
   \[ (x + 1)^2 + (y + 1)^2 \leq 16 \]
10-7 Practice

Solving Linear-Nonlinear Systems

Solve each system of equations.

1. \((x - 2)^2 + y^2 = 5\)
   \[x - y = 1\]

2. \(x = 2(y + 1)^2 - 6\)
   \[x + y = 3\]

3. \(y^2 - 3x^2 = 6\)
   \[y = 2x - 1\]

4. \(x^2 + 2y^2 = 1\)
   \[y = -x + 1\]

5. \(4y^2 - 9x^2 = 36\)
   \[4x^2 - 9y^2 = 36\]

6. \(y = x^2 - 3\)
   \[x^2 + y^2 = 9\]

7. \(x^2 + y^2 = 25\)
   \[4y = 3x\]

8. \(y^2 = 10 - 6x^2\)
   \[4y^2 = 40 - 2x^2\]

9. \(x^2 + y^2 = 25\)
   \[x = 3y - 5\]

10. \(4x^2 + 9y^2 = 36\)
    \[2x^2 - 9y^2 = 18\]

11. \(x = -(y - 3)^2 + 2\)
    \[x = (y - 3)^2 + 3\]

12. \(\frac{x^2}{9} - \frac{y^2}{16} = 1\)
    \[x^2 + y^2 = 9\]

13. \(25x^2 + 4y^2 = 100\)
    \[x = -\frac{5}{2}\]

14. \(x^2 + y^2 = 4\)
    \[\frac{x^2}{4} + \frac{y^2}{8} = 1\]

15. \(x^2 - y^2 = 3\)
    \[y^2 - x^2 = 3\]

16. \(\frac{x^2}{7} + \frac{y^2}{7} = 1\)
    \[3x^2 - y^2 = 9\]

17. \(x + 2y = 3\)
    \[x^2 + y^2 = 9\]

18. \(x^2 + y^2 = 64\)
    \[x^2 - y^2 = 8\]

Solve each system of inequalities by graphing.

19. \(y \geq x^2\)
    \(y > -x + 2\)

20. \(x^2 + y^2 < 36\)
    \(x^2 + y^2 \geq 16\)

21. \(\frac{(y - 3)^2}{16} + \frac{(x + 2)^2}{4} \leq 1\)
    \((x + 1)^2 + (y - 2)^2 \leq 4\)

22. GEOMETRY The top of an iron gate is shaped like half an ellipse with two congruent segments from the center of the ellipse to the ellipse as shown. Assume that the center of the ellipse is at (0, 0). If the ellipse can be modeled by the equation \(x^2 + 4y^2 = 4\) for \(y \geq 0\) and the two congruent segments can be modeled by \(y = \frac{\sqrt{3}}{2}x\) and \(y = -\frac{\sqrt{3}}{2}x\), what are the coordinates of points A and B?
1. **GRAPHIC DESIGN** A graphic designer is drawing an ellipse and a line. The ellipse is drawn so that it appears on top of the line. In order to determine if the line is partially covered by the ellipse, the program solves for simultaneous solutions of the equations of the line and the ellipse. Complete the following table.

<table>
<thead>
<tr>
<th>No. of Intersections</th>
<th>Covered? Y/N</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

2. **ORBITS** Objects in the solar system travel in elliptical orbits where the Sun is one focal point. Earth’s orbit is an ellipse. What is the maximum number of times per orbit that an asteroid also in an elliptical orbit can cross Earth’s orbit?

3. **CIRCLES** An artist is commissioned to complete a painting of only circles. She wants to include all possible ways circles can relate. What are the possible numbers of intersection points between two circles? For each case, sketch two distinct circles that intersect with the corresponding number of points. Explain why more intersections are not possible.

4. **COLLISION AVOIDANCE** An object is traveling along a hyperbola given by the equation \( \frac{x^2}{9} - \frac{y^2}{36} = 1 \). A probe is launched from the origin along a straight-line path. Mission planners want the probe to get closer and closer to the object, but never hit it. There are two straight lines that meet their criteria. What are they?

5. **TANGENTS** An architect wants a straight path to run from the origin of a coordinate plane to the edge of an elliptically shaped patio so that the pathway forms a tangent to the ellipse. The ellipse is given by the equation \( \frac{(x - 6)^2}{12} + \frac{y^2}{96} = 1 \).

   a. Using the equation \( y = mx \) to describe the path, substitute into the equation for the ellipse to get a quadratic equation in \( x \).

   b. Solve for \( m \) in the equation you found for part a when \( x = 4 \).
Trigonometric Functions in Right Triangles

Trigonometric Functions for Acute Angles  Trigonometry is the study of relationships among the angles and sides of a right triangle. A trigonometric function has a rule given by a trigonometric ratio, which is a ratio that compares the side lengths of a right triangle.

<table>
<thead>
<tr>
<th>Trigonometric Functions in Right Triangles</th>
<th>If $\theta$ is the measure of an acute angle of a right triangle, $\text{opp}$ is the measure of the leg opposite $\theta$, $\text{adj}$ is the measure of the leg adjacent to $\theta$, and $\text{hyp}$ is the measure of the hypotenuse, then the following are true.</th>
</tr>
</thead>
</table>
| $\sin \theta = \frac{\text{opp}}{\text{hyp}}$ | $\cos \theta = \frac{\text{adj}}{\text{hyp}}$  
$\tan \theta = \frac{\text{opp}}{\text{adj}}$ |
| $\csc \theta = \frac{\text{hyp}}{\text{opp}}$ | $\sec \theta = \frac{\text{hyp}}{\text{adj}}$  
$\cot \theta = \frac{\text{adj}}{\text{opp}}$ |

Example  
In a right triangle, $\angle B$ is acute and $\cos B = \frac{3}{7}$. Find the value of $\tan B$.

Step 1  Draw a right triangle and label one acute angle $B$. Label the adjacent side 3 and the hypotenuse 7.

Step 2  Use the Pythagorean Theorem to find $b$.

\[
a^2 + b^2 = c^2 \quad \text{Pythagorean Theorem} \\
3^2 + b^2 = 7^2 \quad a = 3 \text{ and } c = 7 \\
9 + b^2 = 49 \quad \text{Simplify.} \\
b^2 = 40 \quad \text{Subtract 9 from each side.} \\
b = \sqrt{40} = 2\sqrt{10} \quad \text{Take the positive square root of each side.}
\]

Step 3  Find $\tan B$.

\[
\tan B = \frac{\text{opp}}{\text{adj}} \quad \text{Tangent function} \\
\tan B = \frac{2\sqrt{10}}{3} \quad \text{Replace opp with } 2\sqrt{10} \text{ and adj with 3.}
\]

Exercises

Find the values of the six trigonometric functions for angle $\theta$.

1. 

2. 

3. 

In a right triangle, $\angle A$ and $\angle B$ are acute.

4. If $\tan A = \frac{7}{12}$, what is $\cos A$? 

5. If $\cos A = \frac{1}{2}$, what is $\tan A$? 

6. If $\sin B = \frac{3}{8}$, what is $\tan B$?
Use Trigonometric Functions You can use trigonometric functions to find missing side lengths and missing angle measures of right triangles. You can find the measure of the missing angle by using the inverse of sine, cosine, or tangent.

Example

Find the measure of $\angle C$. Round to the nearest tenth if necessary.

You know the measure of the side opposite $\angle C$ and the measure of the hypotenuse. Use the sine function.

\[
\sin C = \frac{\text{opp}}{\text{hyp}} \quad \text{Sine function}
\]

\[
\sin C = \frac{8}{10} \quad \text{Replace opp with 8 and hyp with 10.}
\]

\[
\sin^{-1} \frac{8}{10} = m\angle C \quad \text{Inverse sine}
\]

\[
53.1^\circ = m\angle C \quad \text{Use a calculator.}
\]

Exercises

Use a trigonometric function to find each value of $x$. Round to the nearest tenth if necessary.

1. \[\text{\begin{tabular}{c}
38\degree \\
10
\end{tabular}}\]

2. \[\text{\begin{tabular}{c}
63\degree \\
4
\end{tabular}}\]

3. \[\text{\begin{tabular}{c}
14.5 \\
20\degree \\
x
\end{tabular}}\]

4. \[\text{\begin{tabular}{c}
\quad \\
5
\end{tabular}}\]

5. \[\text{\begin{tabular}{c}
8 \\
32\degree \\
x
\end{tabular}}\]

6. \[\text{\begin{tabular}{c}
70\degree \\
x \\
9
\end{tabular}}\]

Find $x$. Round to the nearest tenth if necessary.

7. \[\text{\begin{tabular}{c}
7 \\
4
\end{tabular}}\]

8. \[\text{\begin{tabular}{c}
33 \\
13
\end{tabular}}\]

9. \[\text{\begin{tabular}{c}
x \\
10 \\
4
\end{tabular}}\]
Find the values of the six trigonometric functions for angle $\theta$.

1. $\theta$  
2. $\theta$  
3. $\theta$  

In a right triangle, $\angle A$ and $\angle B$ are acute.

4. If $\tan B = 2$, what is $\cos B$?  
5. If $\tan A = \frac{11}{17}$, what is $\sin A$?  
6. If $\sin B = \frac{8}{15}$, what is $\cos B$?

Use a trigonometric function to find each value of $x$. Round to the nearest tenth if necessary.

7.  
8.  
9.  

Use trigonometric functions to find the values of $x$ and $y$. Round to the nearest tenth if necessary.

10.  
11.  
12.  

13. **SURVEYING** John stands 150 meters from a water tower and sights the top at an angle of elevation of 36°. If John’s eyes are 2 meters above the ground, how tall is the tower? Round to the nearest meter.
1. **ROOFS** The roof on a house is built with a pitch of 10/12, meaning that the roof rises 10 feet for every 12 feet of horizontal run. The side view of the roof is shown in the figure below.

![Roof Diagram](image)

a. What is the angle $x$ at the base of the roof?

b. What is the angle $y$ at the peak of the roof?

c. What is the length $\ell$ of the roof?

d. If the width of the roof is 26 feet, what is the total area of the roof?

2. **BUILDINGS** Jessica stands 150 feet from the base of a tall building. She measures the angle from her eye to the top of the building to be $84^\circ$. If her eye level is 5 feet above the ground, how tall is the building?

3. **SCALE DRAWING** The collection pool for a fountain is in the shape of a right triangle. A scale drawing shows that the angles of the triangle are 40°, 50°, and 90°. If the hypotenuse of the actual fountain will be 30 feet, what are the lengths of the other two sides of the fountain?

4. **GEOMETRY** A regular hexagon is inscribed in a circle with a diameter of 8 inches.

![Hexagon Diagram](image)

a. What is the perimeter of the hexagon?

b. What is the area of the hexagon?
Trigonometric Functions of General Angles

Trigonometric Functions for General Angles

Let \( \theta \) be an angle in standard position and let \( P(x, y) \) be a point on the terminal side of \( \theta \). By the Pythagorean Theorem, the distance \( r \) from the origin is given by \( r = \sqrt{x^2 + y^2} \). The trigonometric functions of an angle in standard position may be defined as follows.

\[
\begin{align*}
\sin \theta &= \frac{y}{r} \\
\cos \theta &= \frac{x}{r} \\
\tan \theta &= \frac{y}{x}, \quad x \neq 0 \\
\csc \theta &= \frac{r}{y}, \quad y \neq 0 \\
\sec \theta &= \frac{r}{x}, \quad x \neq 0 \\
\cot \theta &= \frac{x}{y}, \quad y \neq 0
\end{align*}
\]

Example

Find the exact values of the six trigonometric functions of \( \theta \) if the terminal side of \( \theta \) in standard position contains the point \((-5, 5\sqrt{2})\).

You know that \( x = -5 \) and \( y = 5 \). You need to find \( r \).

\[
r = \sqrt{x^2 + y^2} \\
= \sqrt{(-5)^2 + (5\sqrt{2})^2} \\
= \sqrt{75} \text{ or } 5\sqrt{3}
\]

Now use \( x = -5, y = 5\sqrt{2} \), and \( r = 5\sqrt{3} \) to write the six trigonometric ratios.

\[
\begin{align*}
\sin \theta &= \frac{y}{r} = \frac{5\sqrt{2}}{5\sqrt{3}} = \frac{\sqrt{6}}{3} \\
\cos \theta &= \frac{x}{r} = -\frac{5}{5\sqrt{3}} = -\frac{\sqrt{3}}{3} \\
\tan \theta &= \frac{y}{x} = -\frac{5\sqrt{2}}{-5} = -\sqrt{2} \\
\csc \theta &= \frac{r}{y} = \frac{5\sqrt{3}}{5\sqrt{2}} = \frac{\sqrt{6}}{2} \\
\sec \theta &= \frac{r}{x} = \frac{5\sqrt{3}}{-5} = -\sqrt{3} \\
\cot \theta &= \frac{x}{y} = -\frac{5}{5\sqrt{2}} = -\frac{\sqrt{2}}{2}
\end{align*}
\]

Exercises

The terminal side of \( \theta \) in standard position contains each point. Find the exact values of the six trigonometric functions of \( \theta \).

1. (8, 4)  
2. (4, 4)  
3. (0, 4)  
4. (6, 2)
Trigonometric Functions of General Angles

### Trigonometric Functions with Reference Angles
If $\theta$ is a nonquadrantal angle in standard position, its **reference angle** $\theta'$ is defined as the acute angle formed by the terminal side of $\theta$ and the $x$-axis.

#### Reference Angle Rule

<table>
<thead>
<tr>
<th>Quadrant</th>
<th>Reference Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$\theta'$ = $\theta$</td>
</tr>
</tbody>
</table>
| II       | $\theta'$ = $180^\circ - \theta$  
($\theta'$ = $\pi - \theta$) |
| III      | $\theta'$ = $\theta - 180^\circ$  
($\theta'$ = $\theta - \pi$) |
| IV       | $\theta'$ = $360^\circ - \theta$  
($\theta'$ = $2\pi - \theta$) |

**Example 1**

<table>
<thead>
<tr>
<th>Sketch an angle of measure 205°. Then find its reference angle.</th>
</tr>
</thead>
</table>

Because the terminal side of 205° lies in Quadrant III, the reference angle $\theta'$ is $205^\circ - 180^\circ$ or 25°.

**Example 2**

<table>
<thead>
<tr>
<th>Use a reference angle to find the exact value of $\cos \frac{3\pi}{4}$.</th>
</tr>
</thead>
</table>

Because the terminal side of $\frac{3\pi}{4}$ lies in Quadrant II, the reference angle $\theta'$ is $\pi - \frac{3\pi}{4}$ or $\frac{\pi}{4}$.

The cosine function is negative in Quadrant II.

\[
\cos \frac{3\pi}{4} = -\cos \frac{\pi}{4} = -\frac{\sqrt{2}}{2}.
\]

**Exercises**

Sketch each angle. Then find its reference angle.

1. 155°
2. 230°
3. $\frac{4\pi}{3}$
4. $-\frac{\pi}{6}$

Find the exact value of each trigonometric function.

5. $\tan 330^\circ$
6. $\cos \frac{11\pi}{4}$
7. $\cot 30^\circ$
8. $\csc \frac{\pi}{4}$
13-3 Practice

Trigonometric Functions of General Angles

The terminal side of \( \theta \) in standard position contains each point. Find the exact values of the six trigonometric functions of \( \theta \).

1. \((6, 8)\)  
2. \((-20, 21)\)  
3. \((-2, -5)\)

Sketch each angle. Then find its reference angle.

4. \(\frac{13\pi}{8}\)  
5. \(-210^\circ\)  
6. \(-\frac{7\pi}{4}\)

Find the exact value of each trigonometric function.

7. \(\tan 135^\circ\)  
8. \(\cot 210^\circ\)  
9. \(\cot (-90^\circ)\)  
10. \(\cos 405^\circ\)

11. \(\tan \frac{5\pi}{3}\)  
12. \(\csc \left(-\frac{3\pi}{4}\right)\)  
13. \(\cot 2\pi\)  
14. \(\tan \frac{13\pi}{6}\)

15. LIGHT Light rays that “bounce off” a surface are reflected by the surface. If the surface is partially transparent, some of the light rays are bent or refracted as they pass from the air through the material. The angles of reflection \( \theta_1 \) and of refraction \( \theta_2 \) in the diagram at the right are related by the equation \( \sin \theta_1 = n \sin \theta_2 \). If \( \theta_1 = 60^\circ \) and \( n = \sqrt{3} \), find the measure of \( \theta_2 \).

16. FORCE A cable running from the top of a utility pole to the ground exerts a horizontal pull of 800 Newtons and a vertical pull of \(800\sqrt{3}\) Newtons. What is the sine of the angle \( \theta \) between the cable and the ground? What is the measure of this angle?
13-3 Word Problem Practice

Trigonometric Functions of General Angles

1. RADIOS Two correspondence radios are located 2 kilometers away from a base camp. The angle formed between the first radio, the base camp, and the second radio is 120°. If the first radio has coordinates (2, 0) relative to the base camp, what is the position of the second radio relative to the base camp?

2. CLOCKS The pendulum of a grandfather clock swings back and forth through an arc. The angle $\theta$ of the pendulum is given by $\theta = 0.3 \cos \left( \frac{\pi}{2} + 5t \right)$ where $t$ is the time in seconds after leaving the bottom of the swing. Determine the measure of the angles in radians for $t = 0, 0.5, 1, 1.5, 2, 2.5,$ and $3$ seconds.

3. FERRIS WHEELS Janice rides a Ferris wheel in Japan called the Sky Dream Fukuoka, which has a radius of about 60 m and is 5 m off the ground. After she enters the bottom car, the wheel rotates 210.5° counterclockwise before stopping. How high above the ground is Janice when the car has stopped?

4. SOCCER Alice kicks a soccer ball towards a wall. The ball is deflected off the wall at an angle of 40°, and it travels 6 meters. How far is the soccer ball from the wall when it stops rolling?

5. PAPER AIRPLANES The formula $R = \frac{V_0^2 \sin 2\theta}{32} + 15 \cos \theta$ gives the distance traveled by a paper airplane that is thrown with an initial velocity of $V_0$ feet per second at an angle of $\theta$ with the ground.

   a. If the airplane is thrown with an initial velocity of 15 feet per second at an angle of 25°, how far will the airplane travel?

   b. Two airplanes are thrown with an initial velocity of 10 feet per second. One airplane is thrown at an angle of 15° to the ground, and the other airplane is thrown at an angle of 45° to the ground. Which will travel farther?
**Circular Functions**

<table>
<thead>
<tr>
<th>Definition of Sine and Cosine</th>
<th>If the terminal side of an angle $\theta$ in standard position intersects the unit circle at $P(x, y)$, then $\cos \theta = x$ and $\sin \theta = y$. Therefore, the coordinates of $P$ can be written as $P(\cos \theta, \sin \theta)$.</th>
</tr>
</thead>
</table>

**Example**

The terminal side of angle $\theta$ in standard position intersects the unit circle at $P\left(-\frac{5}{6}, \frac{\sqrt{11}}{6}\right)$. Find $\cos \theta$ and $\sin \theta$.

$$P\left(-\frac{5}{6}, \frac{\sqrt{11}}{6}\right) = P(\cos \theta, \sin \theta),$$

so $\cos \theta = -\frac{5}{6}$ and $\sin \theta = \frac{\sqrt{11}}{6}$.

**Exercises**

The terminal side of angle $\theta$ in standard position intersects the unit circle at each point $P$. Find $\cos \theta$ and $\sin \theta$.

1. $P\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
2. $P(0, -1)$
3. $P\left(-\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$
4. $P\left(-\frac{4}{5}, -\frac{3}{5}\right)$
5. $P\left(\frac{1}{6}, -\frac{\sqrt{35}}{6}\right)$
6. $P\left(\frac{\sqrt{7}}{4}, \frac{3}{4}\right)$
7. $P$ is on the terminal side of $\theta = 45^\circ$.
8. $P$ is on the terminal side of $\theta = 120^\circ$.
9. $P$ is on the terminal side of $\theta = 240^\circ$.
10. $P$ is on the terminal side of $\theta = 330^\circ$. 

---

**Definition of Sine and Cosine**

If the terminal side of an angle $\theta$ in standard position intersects the unit circle at $P(x, y)$, then $\cos \theta = x$ and $\sin \theta = y$. Therefore, the coordinates of $P$ can be written as $P(\cos \theta, \sin \theta)$.
**Circular Functions**

**Periodic Functions**

A periodic function has y-values that repeat at regular intervals. One complete pattern is called a cycle, and the horizontal length of one cycle is called a period.

The sine and cosine functions are periodic; each has a period of 360° or 2π radians.

**Example 1**

Determine the period of the function.

The pattern of the function repeats every 10 units, so the period of the function is 10.

**Example 2**

Find the exact value of each function.

a. \( \sin 855° \)

\[
\sin 855° = \sin (135° + 720°) = \sin 135° = \frac{\sqrt{2}}{2}
\]

b. \( \cos \left(\frac{31\pi}{6}\right) \)

\[
\cos \left(\frac{31\pi}{6}\right) = \cos \left(\frac{7\pi}{6} + 4\pi\right) = \cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}
\]

**Exercises**

Determine the period of each function.

1. [Diagram of a periodic function]

2. [Diagram of a periodic function]

Find the exact value of each function.

3. \( \sin (-510°) \)

4. \( \sin 495° \)

5. \( \cos \left(-\frac{5\pi}{2}\right) \)

6. \( \sin \frac{5\pi}{3} \)

7. \( \cos \frac{11\pi}{4} \)

8. \( \sin \left(-\frac{3\pi}{4}\right) \)
Circular Functions

The terminal side of angle $\theta$ in standard position intersects the unit circle at each point $P$. Find $\cos \theta$ and $\sin \theta$.

1. $P \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$

2. $P \left( \frac{20}{29}, -\frac{21}{29} \right)$

3. $P(0.8, 0.6)$

4. $P(0, -1)$

5. $P \left( -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)$

6. $P \left( \frac{\sqrt{3}}{2}, \frac{1}{2} \right)$

Determine the period of each function.

7. 

8. 

Find the exact value of each function.

9. $\cos \frac{7\pi}{4}$

10. $\sin (-30^\circ)$

11. $\sin \left( -\frac{2\pi}{3} \right)$

12. $\cos (-330^\circ)$

13. $\cos 600^\circ$

14. $\sin \frac{9\pi}{2}$

15. $\cos 7\pi$

16. $\cos \left( -\frac{11\pi}{4} \right)$

17. $\sin (-225^\circ)$

18. $\sin 585^\circ$

19. $\cos \left( -\frac{10\pi}{3} \right)$

20. $\sin 840^\circ$

21. **FERRIS WHEELS** A Ferris wheel with a diameter of 100 feet completes 2.5 revolutions per minute. What is the period of the function that describes the height of a seat on the outside edge of the Ferris wheel as a function of time?
### Circular Functions

**1. TIRES** A point on the edge of a car tire is marked with paint. As the car moves slowly, the marked point on the tire varies in distance from the surface of the road. The height in inches of the point is given by the expression \( h = -8 \cos t + 8 \), where \( t \) is the time in seconds.

**a.** What is the maximum height above ground that the point on the tire reaches?

**b.** What is the minimum height above ground that the point on the tire reaches?

**c.** How many rotations does the tire make per second?

**d.** How far does the marked point travel in 30 seconds? How far does the marked point travel in one hour?

---

**2. GEOMETRY** The temperature \( T \) in degrees Fahrenheit of a city \( t \) months into the year is approximated by the formula \( T = 42 + 30 \sin \left( \frac{\pi}{6} t \right) \).

**a.** What is the highest monthly temperature for the city?

**b.** In what month does the highest temperature occur?

**c.** What is the lowest monthly temperature for the city?

**d.** In what month does the lowest temperature occur?

---

**3. THE MOON** The Moon’s period of revolution is the number of days it takes for the Moon to revolve around Earth. The period can be determined by graphing the percentage of sunlight reflected by the Moon each day, as seen by an observer on Earth. Use the graph to determine the Moon’s period of revolution.

---

*Image of a graph showing the Moon’s orbit with sunlight reflected (%).*
Graphing Trigonometric Functions

Sine, Cosine, and Tangent Functions  Trigonometric functions can be graphed on the coordinate plane. Graphs of periodic functions have repeating patterns, or cycles; the horizontal length of each cycle is the period. The amplitude of the graph of a sine or cosine function equals half the difference between the maximum and minimum values of the function. Tangent is a trigonometric function that has asymptotes when graphed.

**Parent Function**

<table>
<thead>
<tr>
<th>Sine, Cosine, and Tangent Functions</th>
<th>( y = \sin \theta )</th>
<th>( y = \cos \theta )</th>
<th>( y = \tan \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain</td>
<td>(all real numbers)</td>
<td>(all real numbers)</td>
<td>( \theta \mid \theta \neq 90 + 180n, n \text{ is an integer} )</td>
</tr>
<tr>
<td>Range</td>
<td>( y \mid -1 \leq y \leq 1 )</td>
<td>( y \mid -1 \leq y \leq 1 )</td>
<td>( \text{all real numbers} )</td>
</tr>
<tr>
<td>Amplitude</td>
<td>1</td>
<td>1</td>
<td>undefined</td>
</tr>
<tr>
<td>Period</td>
<td>360°</td>
<td>360°</td>
<td>180°</td>
</tr>
</tbody>
</table>

**Example**

Find the amplitude and period of each function. Then graph the function.

a. \( y = 4 \cos \frac{\theta}{3} \)

First, find the amplitude.

\[ |a| = |4|, \text{ so the amplitude is } 4. \]

Next find the period.

\[ \frac{360°}{\frac{1}{3}} = 1080° \]

Use the amplitude and period to help graph the function.

b. \( y = \frac{1}{2} \tan 2\theta \)

The amplitude is not defined, and the period is 90°.

**Exercises**

Find the amplitude, if it exists, and period of each function. Then graph the function.

1. \( y = -4 \sin \theta \)

2. \( y = 2 \tan \frac{\theta}{2} \)
Graphing Trigonometric Functions

Graphs of Other Trigonometric Functions

The graphs of the cosecant, secant, and cotangent functions are related to the graphs of the sine, cosine, and tangent functions.

<table>
<thead>
<tr>
<th>Cosecant, Secant, and Cotangent Functions</th>
<th>Parent Function</th>
<th>$y = \csc \theta$</th>
<th>$y = \sec \theta$</th>
<th>$y = \cot \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain</td>
<td>${ \theta \mid \theta \neq 180n, n \text{ is an integer} }$</td>
<td>${ \theta \mid \theta \neq 90 + 180n, n \text{ is an integer} }$</td>
<td>${ \theta \mid \theta \neq 180n, n \text{ is an integer} }$</td>
<td></td>
</tr>
<tr>
<td>Range</td>
<td>${ y \mid -1 &gt; y \text{ or } y &gt; 1 }$</td>
<td>${ y \mid -1 &gt; y \text{ or } y &gt; 1 }$</td>
<td>${ \text{all real numbers} }$</td>
<td></td>
</tr>
<tr>
<td>Amplitude</td>
<td>undefined</td>
<td>undefined</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>Period</td>
<td>$360^\circ$</td>
<td>$360^\circ$</td>
<td>$180^\circ$</td>
<td></td>
</tr>
</tbody>
</table>

**Example**

Find the period of $y = \frac{1}{2} \csc \theta$. Then graph the function.

Since $\frac{1}{2} \csc \theta$ is a reciprocal of $\frac{1}{2} \sin \theta$, the graphs have the same period, $360^\circ$. The vertical asymptotes occur at the points where $\frac{1}{2} \sin \theta = 0$.

So, the asymptotes are at $\theta = 0^\circ$, $\theta = 180^\circ$, and $\theta = 360^\circ$. Sketch $y = \frac{1}{2} \sin \theta$ and use it to graph $y = \frac{1}{2} \csc \theta$.

**Exercises**

Find the period of each function. Then graph the function.

1. $y = \cot 2\theta$

2. $y = \sec 3\theta$
**13-7 Practice**

**Graphing Trigonometric Functions**

Find the amplitude, if it exists, and period of each function. Then graph the function.

1. \( y = -4 \sin \theta \)  
2. \( y = \cot \frac{1}{2} \theta \)  
3. \( y = \cos 5\theta \)

4. \( y = \csc \frac{3}{4} \theta \)  
5. \( y = 2 \tan \frac{1}{2} \theta \)  
6. \( y = \frac{1}{2} \sin \theta \)

7. **FORCE**  
   An anchoring cable exerts a force of 500 Newtons on a pole. The force has the horizontal and vertical components \( F_x \) and \( F_y \). (A force of one Newton (N), is the force that gives an acceleration of 1 m/sec\(^2\) to a mass of 1 kg.)

   a. The function \( F_x = 500 \cos \theta \) describes the relationship between the angle \( \theta \) and the horizontal force. What are the amplitude and period of this function?

   b. The function \( F_y = 500 \sin \theta \) describes the relationship between the angle \( \theta \) and the vertical force. What are the amplitude and period of this function?

8. **WEATHER**  
   The function \( y = 60 + 25 \sin \frac{\pi t}{6} \), where \( t \) is in months and \( t = 0 \) corresponds to April 15, models the average high temperature in degrees Fahrenheit in Centerville.

   a. Determine the period of this function. What does this period represent?

   b. What is the maximum high temperature and when does this occur?
1. PHYSICS  The following chart gives functions which model the wave patterns of different colors of light emitted from a particular source, where $y$ is the height of the wave in nanometers and $t$ is the length from the start of the wave in nanometers.

<table>
<thead>
<tr>
<th>Color</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td>$y = 300 \sin \left( \frac{\pi}{350} t \right)$</td>
</tr>
<tr>
<td>Orange</td>
<td>$y = 125 \sin \left( \frac{\pi}{305} t \right)$</td>
</tr>
<tr>
<td>Yellow</td>
<td>$y = 460 \sin \left( \frac{\pi}{290} t \right)$</td>
</tr>
<tr>
<td>Green</td>
<td>$y = 200 \sin \left( \frac{\pi}{260} t \right)$</td>
</tr>
<tr>
<td>Blue</td>
<td>$y = 40 \sin \left( \frac{\pi}{235} t \right)$</td>
</tr>
<tr>
<td>Violet</td>
<td>$y = 80 \sin \left( \frac{\pi}{210} t \right)$</td>
</tr>
</tbody>
</table>

a. What are the amplitude and period of the function describing green light waves?

b. The intensity of a light wave corresponds directly to its amplitude. Which color emitted from the source is the most intense?

c. The color of light depends on the period of the wave. Which color has the shortest period? The longest period?

2. SWIMMING  As Charles swims a 25 meter sprint, the position of his right hand relative to the water surface can be modeled by the graph below, where $g$ is the height of the hand in inches from the water level and $t$ is the time in seconds past the start of the sprint. What function describes this graph?

3. ENVIRONMENT  In a certain forest, the leaf density can be modeled by the equation $y = 20 + 15 \sin \left( \frac{\pi}{6} (t - 3) \right)$ where $y$ represents the number of leaves per square foot and $t$ represents the month where January $= 1$.

a. Determine the period of this function. What does this period represent?

b. What is the maximum leaf density that occurs in this forest and when does this occur?
Translations of Trigonometric Graphs

Horizontal Translations When a constant is subtracted from the angle measure in a trigonometric function, a phase shift of the graph results.

The phase shift of the graphs of the functions $y = a \sin b(\theta - h)$, $y = a \cos b(\theta - h)$, and $y = a \tan b(\theta - h)$ is $h$, where $b > 0$. If $h > 0$, the shift is $h$ units to the right. If $h < 0$, the shift is $|h|$ units to the left.

**Example** State the amplitude, period, and phase shift for $y = \frac{1}{2} \cos 3\left(\theta - \frac{\pi}{2}\right)$. Then graph the function.

- **Amplitude:** $|a| = \frac{1}{2}$
- **Period:** $\frac{2\pi}{|b|} = \frac{2\pi}{3}$
- **Phase Shift:** $h = \frac{\pi}{2}$

The phase shift is to the right since $\frac{\pi}{2} > 0$.

**Exercises**

State the amplitude, period, and phase shift for each function. Then graph the function.

1. $y = 2 \sin (\theta + 60^\circ)$
2. $y = \tan \left(\theta - \frac{\pi}{2}\right)$
3. $y = 3 \cos (\theta - 45^\circ)$
4. $y = \frac{1}{2} \sin 3 \left(\theta - \frac{\pi}{3}\right)$
Translations of Trigonometric Graphs

Vertical Translations When a constant is added to a trigonometric function, the graph is shifted vertically.

Vertical Shift

The vertical shift of the graphs of the functions $y = a \sin b(\theta - h) + k$, $y = a \cos b(\theta - h) + k$, and $y = a \tan b(\theta - h) + k$ is $k$.

If $k > 0$, the shift is $k$ units up.

If $k < 0$, the shift is $|k|$ units down.

The midline of a vertical shift is $y = k$.

Graphing Trigonometric Functions

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Determine the vertical shift, and graph the midline.</td>
</tr>
<tr>
<td>2</td>
<td>Determine the amplitude, if it exists. Use dashed lines to indicate the maximum and minimum values of the function.</td>
</tr>
<tr>
<td>3</td>
<td>Determine the period of the function and graph the appropriate function.</td>
</tr>
<tr>
<td>4</td>
<td>Determine the phase shift and translate the graph accordingly.</td>
</tr>
</tbody>
</table>

Example State the amplitude, period, vertical shift, and equation of the midline for $y = \cos 2\theta - 3$. Then graph the function.

Amplitude: $|a| = |1|$ or 1

Period: $\frac{2\pi}{|b|} = \frac{2\pi}{|2|}$ or $\pi$

Vertical Shift: $k = -3$, so the vertical shift is 3 units down.

The equation of the midline is $y = -3$.

Since the amplitude of the function is 1, draw dashed lines parallel to the midline that are 1 unit above and below the midline. Then draw the cosine curve, adjusted to have a period of $\pi$.

Exercises

State the amplitude, period, vertical shift, and equation of the midline for each function. Then graph the function.

1. $y = \frac{1}{2} \cos \theta + 2$

2. $y = 3 \sin \theta - 2$
**Translations of Trigonometric Graphs**

State the amplitude, period, phase shift, and vertical shift for each function. Then graph the function.

1. \(y = \frac{1}{2} \tan \left( \theta - \frac{\pi}{2} \right)\)
2. \(y = 2 \cos (\theta + 30^\circ) + 3\)
3. \(y = 3 \csc (2\theta + 60^\circ) - 2.5\)
4. \(y = -3 + 2 \sin 2\left(\theta + \frac{\pi}{4}\right)\)
5. \(y = 3 \cos 2(\theta + 45^\circ) + 1\)
6. \(y = -1 + 4 \tan (\theta + \pi)\)

7. **ECOLOGY** The population of an insect species in a stand of trees follows the growth cycle of a particular tree species. The insect population can be modeled by the function \(y = 40 + 30 \sin 6t\), where \(t\) is the number of years since the stand was first cut in November, 1920.

a. How often does the insect population reach its maximum level?

b. When did the population last reach its maximum?

c. What condition in the stand do you think corresponds with a minimum insect population?
Word Problem Practice

Translations of Trigonometric Graphs

1. CLOCKS A town hall has a tower with a clock on its face. The center of the clock is 40 feet above street level. The minute hand of the clock has a length of four feet.

   a. What is the maximum height of the tip of the minute hand above street level?

   b. What is the minimum height of the tip of the minute hand above street level?

   c. Write a sine function that represents the height above street level of the tip of the minute hand for $t$ minutes after midnight.

   d. Graph the function from your answer to part c.

2. ANIMAL POPULATION The population of predators and prey in a closed ecological system tends to vary periodically over time. In a certain system, the population of snakes $S$ can be represented by $S = 100 + 20 \sin \left(\frac{\pi}{5}t\right)$, where $t$ is the number of years since January 1, 2000. In that same system, the population of rats can be represented by $R = 200 + 75 \sin \left(\frac{\pi}{5}t + \frac{\pi}{10}\right)$.

   a. What is the maximum snake population?

   b. When is this population first reached?

   c. What is the minimum rat population?

   d. When is this population first reached?
Inverse Trigonometric Functions

If you know the value of a trigonometric function for an angle, you can use the inverse to find the angle. If you restrict the function’s domain, then the inverse is a function. The values in this restricted domain are called principal values.

<table>
<thead>
<tr>
<th>Principal Values of Sine, Cosine, and Tangent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \sin x ) if and only if ( y = \sin x ) and ( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} ).</td>
</tr>
<tr>
<td>( y = \cos x ) if and only if ( y = \cos x ) and ( 0 \leq x \leq \pi ).</td>
</tr>
<tr>
<td>( y = \tan x ) if and only if ( y = \tan x ) and ( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} ).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Inverse Sine, Cosine, and Tangent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given ( y = \sin x ), the inverse sine function is defined by ( y = \sin^{-1} x ) or ( y = \arcsin x ).</td>
</tr>
<tr>
<td>Given ( y = \cos x ), the inverse cosine function is defined by ( y = \cos^{-1} x ) or ( y = \arccos x ).</td>
</tr>
<tr>
<td>Given ( y = \tan x ), the inverse tangent function is given by ( y = \tan^{-1} x ) or ( y = \arctan x ).</td>
</tr>
</tbody>
</table>

Example 1

Find the value of \( \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \). Write angle measures in degrees and radians.

Find the angle \( \theta \) for \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \) that has a sine value of \( \frac{\sqrt{3}}{2} \).

Using a unit circle, the point on the circle that has \( y \)-coordinate of \( \frac{\sqrt{3}}{2} \) is \( \frac{\pi}{3} \) or 60°.

So, \( \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \) or 60°.

Example 2

Find \( \tan\left(\sin^{-1}\frac{1}{2}\right) \). Round to the nearest hundredth.

Let \( \theta = \sin^{-1}\frac{1}{2} \). Then \( \sin \theta = \frac{1}{2} \) with \( -\frac{\pi}{2} < \theta < \frac{\pi}{2} \). The value \( \theta = \frac{\pi}{6} \) satisfies both conditions. \( \tan \frac{\pi}{6} = \frac{\sqrt{3}}{3} \) so \( \tan\left(\sin^{-1}\frac{1}{2}\right) = \frac{\sqrt{3}}{3} \).

Exercises

Find each value. Write angle measures in degrees and radians.

1. \( \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \)
2. \( \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) \)
3. \( \arccos\left(-\frac{1}{2}\right) \)
4. \( \arctan \sqrt{3} \)
5. \( \arccos\left(-\frac{\sqrt{2}}{2}\right) \)
6. \( \tan^{-1}(-1) \)

Find each value. Round to the nearest hundredth if necessary.

7. \( \cos\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right] \)
8. \( \tan\left[\arcsin\left(-\frac{5}{7}\right)\right] \)
9. \( \sin\left(\tan^{-1}\frac{5}{12}\right) \)
10. \( \cos\left[\arcsin\left(-0.7\right)\right] \)
11. \( \cos\left(\arctan 5\right) \)
12. \( \sin\left(\cos^{-1} 0.3\right) \)
Inverse Trigonometric Functions

Solve Equations by Using Inverses  You can rewrite trigonometric equations to solve for the measure of an angle.

Example  Solve the equation \( \sin \theta = -0.25 \). Round to the nearest tenth if necessary.

The sine of angle \( \theta \) is \(-0.25\). This can be written as \( \arcsin (-0.25) = \theta \).

Use a calculator to solve.

KEYSTROKES: \( \boxed{2nd} \ [\sin^{-1}] (-) 0.25 \) ENTER \(-14.47751219\)

So, \( \theta \approx -14.5^\circ \)

Exercises

Solve each equation. Round to the nearest tenth if necessary.

1. \( \sin \theta = 0.8 \)
2. \( \tan \theta = 4.5 \)
3. \( \cos \theta = 0.5 \)
4. \( \cos \theta = -0.95 \)
5. \( \sin \theta = -0.1 \)
6. \( \tan \theta = -1 \)
7. \( \cos \theta = 0.52 \)
8. \( \cos \theta = -0.2 \)
9. \( \sin \theta = 0.35 \)
10. \( \tan \theta = 8 \)
Inverse Trigonometric Functions

Find each value. Write angle measures in degrees and radians.

1. \( \text{Arccos} \left( \frac{-1}{2} \right) \)  
2. \( \text{Cos}^{-1} \left( \frac{-\sqrt{2}}{2} \right) \)  
3. \( \text{Tan}^{-1} \left( \frac{-\sqrt{3}}{3} \right) \)

4. \( \text{Arccos} \left( \frac{\sqrt{2}}{2} \right) \)  
5. \( \text{Arctan} \left( -\sqrt{3} \right) \)  
6. \( \text{Sin}^{-1} \left( -\frac{1}{2} \right) \)

Find each value. Round to the nearest hundredth if necessary.

7. \( \tan \left( \text{Cos}^{-1} \left( \frac{1}{2} \right) \right) \)  
8. \( \cos \left[ \text{Sin}^{-1} \left( -\frac{3}{5} \right) \right] \)  
9. \( \cos \left[ \text{Arctan} \left( -1 \right) \right] \)

10. \( \tan \left( \text{Sin}^{-1} \left( \frac{12}{13} \right) \right) \)  
11. \( \sin \left( \text{Arctan} \left( \frac{\sqrt{3}}{3} \right) \right) \)  
12. \( \cos \left( \text{Arctan} \left( \frac{3}{4} \right) \right) \)

Solve each equation. Round to the nearest tenth if necessary.

13. \( \tan \theta = 10 \)  
14. \( \sin \theta = 0.7 \)  
15. \( \sin \theta = -0.5 \)

16. \( \cos \theta = 0.05 \)  
17. \( \tan \theta = 0.22 \)  
18. \( \sin \theta = -0.03 \)

19. **PULLEYS** The equation \( \cos \theta = 0.95 \) describes the angle through which pulley \( A \) moves, and \( \cos \theta = 0.17 \) describes the angle through which pulley \( B \) moves. Which pulley moves through a greater angle?

20. **FLYWHEELS** The equation \( \tan \theta = 1 \) describes the counterclockwise angle through which a flywheel rotates in 1 millisecond. Through how many degrees has the flywheel rotated after 25 milliseconds?
1. **DOORS** The exit from a restaurant kitchen has a pair of swinging doors that meet in the middle of the doorway. Each door is three feet wide. A waiter needs to take a cart of plates into the dining area from the kitchen. The cart is two feet wide.

   ![Diagram of swinging doors and cart]

   **a.** What is the minimum angle \( \theta \) through which the doors must each be opened to prevent the cart from hitting either door?

   **b.** If only one of the two doors could be opened, what is the minimum angle \( \theta \) through which the door must be opened to prevent the cart from hitting the door?

   **c.** If the pair of swinging doors were replaced by a single door the full width of the opening, what is the minimum angle \( \theta \) through which the door must be opened to prevent the cart from hitting the door?

2. **SURVEYING** In ancient times, it was known that a triangle with side lengths of 3, 4, and 5 units was a right triangle. Surveyors used ropes with knots at each unit of length to make sure that an angle was a right angle. Such a rope was placed on the ground so that one leg of the triangle had three knots and the other had four. This guaranteed that the triangle formed was a right triangle, meaning that the surveyor had formed a right angle.

   ![Diagram of surveying rope forming a right triangle]

   To the nearest degree, what are the angle measures in a triangle formed in this way?

3. **TRAVEL** Beth is riding her bike to her friend Marco’s house. She can only ride on the streets, which run north-south or east-west.

   **a.** Beth rides two miles east and four miles south to get to Marco’s. If Beth could have traveled directly from her house to Marco’s, in what direction would she have traveled?

   **b.** Beth then rides three miles west and one mile north to get to the grocery store. If Beth could have traveled directly from Marco’s house to the store, in what direction would she have traveled?
Appropriateness of Linear Models

Sum of the Squared Errors  The sum of the squared errors (SSE) is used to evaluate the appropriateness of a linear model for a set of data. The sum of the squared errors is the sum of the squared residuals. A residual is the difference between an actual y-value and the predicted y-value on the regression line.

Example  The table below shows the number of customers serviced by a lawn company for several years. Calculate the sum of the squared errors.

<table>
<thead>
<tr>
<th>Year</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Customers</td>
<td>13</td>
<td>24</td>
<td>47</td>
<td>68</td>
<td>81</td>
<td>88</td>
<td>109</td>
<td>126</td>
</tr>
</tbody>
</table>

Step 1  Make a scatter plot of the data and calculate the least squares regression line. Determine if the linear model could be appropriate.

The least squares regression equation is about \( y = 16.1x - 19.0 \), where \( x \) represents the number of years since 2000 and \( y \) is the number of customers. The scatter plot resembles a straight line, so this model could be appropriate.

Step 2  Set up a table of values to calculate SSE. The y-value that is predicted by the regression equation is represented by \( yp \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( yp )</th>
<th>( y - yp )</th>
<th>( (y - yp)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>13</td>
<td>13.2</td>
<td>-0.2</td>
<td>0.04</td>
</tr>
<tr>
<td>3</td>
<td>24</td>
<td>29.3</td>
<td>-5.3</td>
<td>28.09</td>
</tr>
<tr>
<td>4</td>
<td>47</td>
<td>45.4</td>
<td>1.6</td>
<td>2.56</td>
</tr>
<tr>
<td>5</td>
<td>68</td>
<td>61.5</td>
<td>6.5</td>
<td>42.25</td>
</tr>
<tr>
<td>6</td>
<td>81</td>
<td>77.6</td>
<td>3.4</td>
<td>11.56</td>
</tr>
<tr>
<td>7</td>
<td>88</td>
<td>93.7</td>
<td>-5.7</td>
<td>32.49</td>
</tr>
<tr>
<td>8</td>
<td>109</td>
<td>109.8</td>
<td>-0.8</td>
<td>0.64</td>
</tr>
<tr>
<td>9</td>
<td>126</td>
<td>125.9</td>
<td>0.1</td>
<td>0.01</td>
</tr>
</tbody>
</table>

\[ \text{SSE} = 117.64 \]

Exercise  The table below shows the number of students that failed the state test on the first attempt each year. Make a scatter plot of the data and calculate the regression equation. Then calculate the sum of the squared errors. Let \( x \) represent the number of years since 2000.

<table>
<thead>
<tr>
<th>Year</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Students</td>
<td>112</td>
<td>93</td>
<td>76</td>
<td>51</td>
<td>43</td>
<td>29</td>
<td>18</td>
</tr>
</tbody>
</table>
Coefficient of Determination  As a percentage, the coefficient of determination provides the likelihood that the related regression line will make an accurate prediction of the data. The formula for the coefficient of determination is \( r^2 = 1 - \frac{\text{SSE}}{\text{SST}} \), where SSE is the sum of squared errors and SST is the total sum of squares. The total sum of squares (SST) is the sum of the squared differences between each y-value and the average y-value.

Example  An ice cream vendor compared the daily sales with the high temperature. Calculate the coefficient of determination and determine if a linear model is appropriate.

Step 1  Make a scatter plot of the data and calculate the least squares regression line. Determine if the linear model could be appropriate. The least squares regression equation is about \( y = 49.58x - 2889.35 \). The scatter plot resembles a straight line, so this model could be appropriate.

Step 2  Set up a table of values to calculate SSE and SST. The predicted y-value is \( \hat{y} \) and the average of the actual y-values is \( \bar{y} \). Calculate and analyze \( r^2 \).

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
x & y & \hat{y} & y - \hat{y} & (y - \hat{y})^2 & y - \bar{y} & (y - \bar{y})^2 \\
\hline
84 & 1225 & 1275.37 & -50.37 & 2537.14 & 1416.17 & -191.17 & 36,545.97 \\
88 & 1417 & 1473.69 & -56.69 & 3213.76 & 1416.17 & 0.83 & 0.69 \\
86 & 1394 & 1374.53 & 19.47 & 379.08 & 1416.17 & -22.17 & 491.51 \\
83 & 1260 & 1225.79 & 34.21 & 1170.32 & 1416.17 & -156.17 & 24,389.07 \\
89 & 1603 & 1523.27 & 79.73 & 6356.87 & 1416.17 & 186.83 & 34,905.45 \\
91 & 1598 & 1622.43 & -24.43 & 596.82 & 1416.17 & 181.83 & 33,062.15 \\
\hline
\end{array}
\]

\[
\text{SSE} = 14,253.99 \quad \text{SST} = 129,394.84
\]

\[
r^2 = 1 - \frac{14,253.99}{129,394.84} \text{ or about 0.89}
\]

The coefficient of determination is about 0.89, so the regression equation is about 89% likely to accurately predict the data. Thus the linear model is appropriate.

Exercise  The adjusted gross domestic product in billions of dollars is shown in the table. The amounts are compared to those in 2000. Calculate the coefficient of determination, and determine if a linear model is appropriate.

<table>
<thead>
<tr>
<th>Year</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDP ($)</td>
<td>42,809</td>
<td>43,969</td>
<td>48,536</td>
<td>53,016</td>
<td>53,249</td>
<td>51,955</td>
</tr>
</tbody>
</table>
Appropriateness of Linear Models

For each exercise, analyze the appropriateness of the linear model.

a. Make a scatter plot of the data and determine whether the relationship between the x-values and y-values could be linear.
b. Identify the least squares regression line, rounding to the nearest thousandth.
c. Find the sum of the squared errors.
d. Find the total sum of squares.
e. Calculate the coefficient of determination.
f. Determine whether the linear model is appropriate. Explain your reasoning.

1. EXERCISE Marisa began exercising with a rowing machine. She kept track of her progress for several weeks.

<table>
<thead>
<tr>
<th>Average Rowing Time Per Session</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
</tbody>
</table>

| Rowing Time (minutes) | 4.3 | 5.1 | 5.7 | 6.4 | 6.8 | 7.3 | 7.4 | 7.9 |

2. RETAIL The table below gives the sales of jeans at a department store chain since 2004. Let x represent the number of years since 2000.

<table>
<thead>
<tr>
<th>Jeans Sales By Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year</td>
</tr>
<tr>
<td>2004</td>
</tr>
<tr>
<td>2005</td>
</tr>
<tr>
<td>2006</td>
</tr>
<tr>
<td>2007</td>
</tr>
<tr>
<td>2008</td>
</tr>
<tr>
<td>2009</td>
</tr>
</tbody>
</table>

| Sales (millions of dollars) | 6.8 | 7.6 | 10.9 | 15.4 | 17.6 | 21.2 |

3. MARATHON The Boston Marathon has been run each year since 1897. The number of entrants in several years are shown. Let x represent the number of years since 1975.

<table>
<thead>
<tr>
<th>Boston Marathon Entrants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year</td>
</tr>
<tr>
<td>1975</td>
</tr>
<tr>
<td>1980</td>
</tr>
<tr>
<td>1985</td>
</tr>
<tr>
<td>1990</td>
</tr>
<tr>
<td>1995</td>
</tr>
<tr>
<td>2000</td>
</tr>
<tr>
<td>2005</td>
</tr>
<tr>
<td>2009</td>
</tr>
</tbody>
</table>

| Entrants | 2395 | 5417 | 5594 | 9412 | 9416 | 17,813 | 20,453 | 26,331 |

4. MAID SERVICE The manager of a maid service kept track of the average amount of time it took her employees to clean a house.

<table>
<thead>
<tr>
<th>Average Cleaning Times</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size of House (ft²)</td>
</tr>
<tr>
<td>900</td>
</tr>
</tbody>
</table>

| Time (minutes) | 63 | 78 | 94 | 106 | 122 | 141 | 158 | 172 |
1. **BIRTHS** The table below shows the total number of births in the United States for selected years.

<table>
<thead>
<tr>
<th>Year</th>
<th>Births (thousands)</th>
<th>Year</th>
<th>Births (thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1985</td>
<td>3761</td>
<td>2002</td>
<td>4022</td>
</tr>
<tr>
<td>1990</td>
<td>4158</td>
<td>2003</td>
<td>4090</td>
</tr>
<tr>
<td>1995</td>
<td>3900</td>
<td>2004</td>
<td>4112</td>
</tr>
<tr>
<td>1999</td>
<td>3959</td>
<td>2005</td>
<td>4140</td>
</tr>
<tr>
<td>2000</td>
<td>4059</td>
<td>2006</td>
<td>4317</td>
</tr>
<tr>
<td>2001</td>
<td>4026</td>
<td>2007</td>
<td>4265</td>
</tr>
</tbody>
</table>

Source: Statistical Abstract of the United States

a. Make a scatter plot of the data and determine whether the relationship between the $x$-values and $y$-values could be linear. Let $x$ represent the number of years since 1985.

b. Calculate SSE and SST.

c. Calculate the coefficient of determination. Determine whether the linear model is appropriate, and explain your reasoning.

2. **EDUCATION** The table below shows the total expenditures by state governments on education.

<table>
<thead>
<tr>
<th>Year</th>
<th>Amount ($ million)</th>
<th>Year</th>
<th>Amount ($ million)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990</td>
<td>7253</td>
<td>2002</td>
<td>16,589</td>
</tr>
<tr>
<td>1995</td>
<td>10,042</td>
<td>2003</td>
<td>17,727</td>
</tr>
<tr>
<td>1999</td>
<td>12,294</td>
<td>2004</td>
<td>19,632</td>
</tr>
<tr>
<td>2000</td>
<td>14,077</td>
<td>2005</td>
<td>20,632</td>
</tr>
<tr>
<td>2001</td>
<td>14,936</td>
<td>2006</td>
<td>20,623</td>
</tr>
</tbody>
</table>

Source: Statistical Abstract of the United States

a. Make a scatter plot of the data and determine whether the relationship between the $x$-values and $y$-values could be linear. Let $x$ represent the number of years since 1990.

b. Calculate SSE and SST.

c. Calculate the coefficient of determination. Determine whether the linear model is appropriate, and explain your reasoning.
Diagnostic Test

1 Which is an equation of a circle with center at (4, -2) and a radius of 6?
   A  $(x - 4)^2 + (y + 2)^2 = 36$
   B  $(x + 4)^2 + (y - 2)^2 = 36$
   C  $(x - 4)^2 + (y + 2)^2 = 6$
   D  $(x + 4)^2 + (y - 2)^2 = 6$

2 Tamara graphs the function $f(x) = \log_2 x$ as shown below.

She then reflects the graph of $f(x)$ over the line $y = x$. What is the equation of the reflected graph?
   A  $g(x) = \sqrt{x}$
   B  $g(x) = 2^x$
   C  $g(x) = \log_2 x$
   D  $g(x) = \frac{1}{\log_2 x}$

3 Which of the following numbers is a real number?
   A  $(-2)^\frac{1}{3}$
   B  $(-3)^\frac{1}{3}$
   C  $\left(-\frac{1}{3}\right)^\frac{1}{2}$
   D  $\left(-\frac{1}{2}\right)^\frac{1}{2}$

4 Parallel lines $j$ and $k$ are cut by a transversal in the figure below.

Which statement must be true?
   A  $\angle 1$ and $\angle 3$ are congruent.
   B  $\angle 2$ and $\angle 4$ are supplementary.
   C  $\angle 1$ and $\angle 4$ are congruent.
   D  $\angle 3$ and $\angle 4$ are supplementary.

5 Solve $\log x - \log 10x^3 = -7$ for $x$.
   A  $x = 0.18$
   B  $x = 3.5$
   C  $x = 316$
   D  $x = 1000$

Go on
6. Tomi created the graph shown below. Which set of steps best describes the transformation of the graph of \( f(x) = x^2 \) into the graph that Tomi created?

A. a reflection over the x-axis, a vertical shrink by a factor of 3, a shift of 1 unit right and 5 units down
B. a reflection over the x-axis, a vertical stretch by a factor of 3, a shift 1 unit right and 5 units down
C. a vertical shrink by a factor of 3, a shift 1 unit right and 5 units down
D. a vertical stretch by a factor of 3, a shift 1 unit right and 5 units down

7. \( \triangle QRS \sim \triangle TUV \), \( QR = 10 \), \( RS = 6 \), \( QS = 14 \), and \( UV = 4 \). What is the perimeter of \( \triangle TUV \)?

A. 45
B. 30
C. 20
D. 12

8. Residents of North Carolina can buy lifetime coastal fishing licenses, with prices determined based on the age of the applicant. The table below shows the number of lifetime coastal fishing licenses sold over a three-day period.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>1 to 11</th>
<th>12 to 64</th>
<th>65 plus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day 1</td>
<td>8</td>
<td>17</td>
<td>5</td>
</tr>
<tr>
<td>Day 2</td>
<td>14</td>
<td>22</td>
<td>8</td>
</tr>
<tr>
<td>Day 3</td>
<td>18</td>
<td>25</td>
<td>6</td>
</tr>
</tbody>
</table>

The following matrix equation can be used to calculate the price in dollars of a license for each age range.

\[
\begin{bmatrix}
8 & 17 & 5 \\
14 & 22 & 8 \\
18 & 25 & 6
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
5600 \\
7840 \\
9130
\end{bmatrix}
\]

Using the matrix equation, what is the price of a lifetime coastal fishing license for an 18 year old applicant?

A. $30
B. $64
C. $150
D. $250

9. Simplify \((32b^3)^{\frac{1}{2}}\).

A. 8b
B. 2b(2b)^{\frac{1}{3}}
C. 2b\sqrt{2b}
D. 16b(2b)^{\frac{1}{2}}
10. Which set of statements illustrates the structure of inductive reasoning?

- Thomas Wolfe was born in Asheville. Asheville is located in western North Carolina. Therefore, Thomas Wolfe was born in western North Carolina.
- Many gardeners living in North Carolina grow the native plant, foamflower. Tim is a gardener who lives in North Carolina. Therefore, Tim grows foamflower in his garden.
- The 17th president of the United States was born in North Carolina. Andrew Johnson was the 17th president. Therefore Andrew Johnson was born in North Carolina.
- Many tourists visit Cape Hatteras Lighthouse located in Buxton, North Carolina. Sally visits Cape Hatteras Lighthouse. Therefore, Sally is in Buxton.

12. A container shaped like a cone holds 245 cubic centimeters of water. How many cubic centimeters of water can a cylindrical container with the same radius and height hold?

- 81.7 cm³
- 122.5 cm³
- 490 cm³
- 735 cm³

13. The length of a football field is 100 yards. Audrey starts at one end of the field and walks back and forth in a straight line from one end of the field to another. The situation is modeled by the graph below, where x represents time in minutes and y represents distance in yards from her starting point.

At what rate is Audrey walking?

- 20 yards per minute
- 50 yards per minute
- 100 yards per minute
- 500 yards per minute
14 Janice draws isosceles triangle $QRS$ with vertex angle $R$ and median $RT$ as shown below.

Which argument can Janice use to prove that $RT$ is also an altitude of $\triangle QRS$?

- **A** Base angles and legs are congruent, so $\angle Q \cong \angle S$ and $QR \cong RS$. $RT \cong RT$ by the reflexive property. $\triangle QTR \cong \triangle STR$ by SSA. Corresponding parts of congruent triangles are congruent, so $\angle RTQ \cong \angle RTS$. Supplementary angles are congruent, so $RT \perp QS$. Therefore $RT$ is an altitude.

- **B** Since the median bisects $QS$ at a right angle, $RT \perp QS$. Therefore $RT$ is an altitude.

- **C** $QT \cong TS$ by the definition of a median. $\triangle QTR \cong \triangle STR$ by SSS. Corresponding parts of congruent triangles are congruent, so $\angle RTQ \cong \angle RTS$. Supplementary angles are congruent, so $RT \perp QS$. Therefore $RT$ is an altitude.

- **D** $QR \cong RS$ since legs are congruent. $\angle RTQ \cong \angle RTS$ since they are opposite congruent sides. Supplementary angles are congruent, so $RT \perp QS$. Therefore $RT$ is an altitude.

15 A store sells three models of a motor scooter at a discount during June and July. Matrix $N$ shows the number of each model sold during each month. Matrix $S$ shows the regular price and the sale price for each model.

$$N = \begin{bmatrix} 61 & 45 & 39 \\ 38 & 29 & 40 \end{bmatrix}$$

$$S = \begin{bmatrix} 1699 & 999 \\ 1895 & 1199 \\ 2195 & 1399 \end{bmatrix}$$

Matrix $P$ is the product of $N$ and $S$, as shown below.

$$P = N \times S = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Which of the following strategies can be used to find the total amount of discounts on all models for the month of June?

- **A** Subtract $P_{11}$ from $P_{12}$.
- **B** Subtract $P_{12}$ from $P_{11}$.
- **C** Subtract $P_{21}$ from $P_{11}$.
- **D** Subtract $P_{22}$ from $P_{12}$.
Diagnostic Test (continued)

16 The table below shows the relationship between the number of pages in each chapter of a book and the number of spelling errors in the chapter.

<table>
<thead>
<tr>
<th>Pages</th>
<th>2</th>
<th>23</th>
<th>22</th>
<th>24</th>
<th>23</th>
<th>17</th>
<th>24</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Errors</td>
<td>27</td>
<td>45</td>
<td>59</td>
<td>23</td>
<td>28</td>
<td>57</td>
<td>59</td>
<td>43</td>
</tr>
</tbody>
</table>

Using the sum of the squared errors from the least-squares regression line for the data, which statement best describes the appropriateness of a linear model for this relationship?

A The sum of the squared errors is 105.37, so a linear model is appropriate.

B The sum of the squared errors is 105.37, so a linear model is not appropriate.

C The sum of the squared errors is 1757.83, so a linear model is appropriate.

D The sum of the squared errors is 1757.83, so a linear model is not appropriate.

17 Which equation is equivalent to \( \log_{4} \frac{1}{16} = x? \)

A \( 4^x = \frac{1}{16} \)

B \( \left( \frac{1}{16} \right)^x = 4 \)

C \( 4 = 16x \)

D \( 4^{\frac{1}{16}} = x \)

18 Add \( \frac{1}{4x} + \frac{x}{x^2 - 4} \).

A \( \frac{5}{4x}; x \neq -2, x \neq 0, \text{ and } x \neq 2 \)

B \( \frac{1 + x}{x^2 + 4x - 4}; x \neq -2, x \neq 0, \text{ and } x \neq 2 \)

C \( \frac{5x^2 - 4}{4x(x^2 - 4)}; x \neq -2, x \neq 0, \text{ and } x \neq 2 \)

D \( \frac{x^2 + 4x - 4}{4x(x^2 - 4)}; x \neq -2, x \neq 0, \text{ and } x \neq 2 \)

19 Troy buys an ice cream cone at the concession stand. He can select vanilla, chocolate, or strawberry, and he can choose either 1, 2, or 3 toppings. Which sample space represents Troy’s choices?

A \( (V, C), (V, S), (C, S), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2) \)

B \( (V, 1), (V, 2), (V, 3), (C, 1), (C, 2), (C, 3), (S, 1), (S, 2), (S, 3) \)

C \( (V, 1), (C, 2), (S, 3), (V, 1), (C, 2), (S, 3), (V, 1), (C, 2), (S, 3) \)

D \( (V, C), (V, S), (S, C), (V, 1), (V, 2), (V, 3), (C, 1), (C, 2), (C, 3), (S, 1), (S, 2), (S, 3) \)
20. Triangle $EFG$ is transformed into triangle $E'F'G'$ as shown below.

The coordinate matrix for $EFG$ is multiplied by matrix $T$, resulting in the coordinate matrix for $E'F'G'$. What is matrix $T$?

A. $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B. $T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

C. $T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

D. $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

21. Quadrilateral $FGHJ$ below is a parallelogram.

What must be true to prove that $FGHJ$ is a rectangle?

A. $\overline{FJ} \parallel \overline{GH}$ and $\overline{FG} \parallel \overline{HJ}$

B. $\overline{FH} \cong \overline{GJ}$

C. $\overline{GL} \cong \overline{LJ}$ and $\overline{FL} \cong \overline{LH}$

D. $\angle HGJ \cong \angle FJG$

22. In 1903, the Wright brothers took the first controlled-power flight at Kitty Hawk. Two propellers pushed their plane as they completed their 12-second, 120-foot flight. Each propeller was 8 feet in diameter and rotated by a chain-and-sprocket transmission system. If the plane was not moving and the edge of the propeller was 2 inches above the ground at its lowest point, how far above the ground would that same edge be, after the propeller rotated 990° counterclockwise?

A. 2 feet 2 inches

B. 4 feet 2 inches

C. 6 feet 2 inches

D. 8 feet 2 inches

23. A metronome’s pendulum completes a cycle every 2 seconds. It has a center point of zero and swings a total distance of 18 centimeters. At $t = 0$, the pendulum is at equilibrium and is starting to swing to the right. Which equation describes the motion of the pendulum cycle?

A. $y = 9 \sin \pi t$

B. $y = 9 \sin \frac{\pi}{2} t$

C. $y = 18 \sin \pi t$

D. $y = 18 \sin \frac{\pi}{2} t$
Diagnostic Test (continued)

24 Which statement supports the Pythagorean Theorem?

A The area of the triangle is \( \frac{1}{2}xy \).
B The area of the smallest square is \( \frac{1}{3} \) the area of the largest square.
C The area of the largest square is equal to the sum of the areas of the smaller two squares.
D The area of the largest square is greater than the sum of the areas of the smaller two squares.

25 A ball is released from the top of a ramp. Which of the following variables would most likely have a strong negative correlation with the time \( t \) after the ball is released?

A the speed \( s \) of the ball
B the height \( h \) of the ball above the ground
C the distance \( d \) traveled by the ball
D the weight \( w \) of the ball

26 Felix graphed a function as shown below.

Which type of function did Felix graph?

A linear
B quadratic
C radical
D absolute value

27 Given: \( \overrightarrow{QR} \) and \( \overrightarrow{RS} \) are secants to circle \( O \), \( m\overset{\frown}{QS} = 87^\circ \), and \( m\overset{\frown}{TU} = 23^\circ \).

What is \( m\angle QRS \)?

A 26°
B 32°
C 55°
D 64°
28. Colin finds the median-fit line for the data in the table below.

<table>
<thead>
<tr>
<th>x</th>
<th>3</th>
<th>7</th>
<th>12</th>
<th>10</th>
<th>9</th>
<th>16</th>
<th>5</th>
<th>14</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>23</td>
<td>35</td>
<td>53</td>
<td>40</td>
<td>27</td>
<td>56</td>
<td>17</td>
<td>32</td>
<td>19</td>
</tr>
</tbody>
</table>

Which ordered pairs most likely represents the three median points he used to find the median-fit line?

A. (5, 17), (9, 27), and (14, 32)
B. (5, 19), (9, 32), and (14, 53)
C. (5, 23), (9, 27), and (14, 53)
D. (7, 35), (9, 27), and (14, 32)

29. In the figure below, \( h \) represents the height of the tree, and \( \sin x = 0.5 \).

\[ \text{How tall is the tree?} \]

A. 16 ft  
B. 30 ft  
C. 60 ft  
D. 64 ft

30. Given:
\[ f(x) = 3(x - 1)^2 \]
\[ g(x) = x^2 - 2x + 5 \]

Which equation represents \( (f - g)(x) \)?

A. \( (f - g)(x) = 2(x^2 - 2x - 1) \)
B. \( (f - g)(x) = 4(x^2 - 2x + 2) \)
C. \( (f - g)(x) = -2(x^2 - 2x - 1) \)
D. \( (f - g)(x) = 2(x^2 - 4x - 1) \)

31. The graph below shows the feasible region for the production of ballasts and wickets.

The profit on a ballast is $7. The profit on a wicket is $3. Which point represents the maximum profit, given the constraints?

A. E  
B. F  
C. G  
D. H
Bethany won the Geography Bee trophy. The globe has a diameter of 12 inches, and the dimensions of the base are shown in the diagram below.

What is the approximate volume of the trophy?

A 864 in$^3$
B 905 in$^3$
C 1015 in$^3$
D 1769 in$^3$

Which expression is equivalent to

\[
\frac{x - 3}{x - 2} + \frac{x}{x^2 - 5x + 6} - \frac{x}{x - 3}
\]

A \( \frac{3}{x - 2} \);
B \( \frac{-x + 9}{x^2 - 5x + 6} \);
C \( \frac{x - 3}{x^2 - 5x + 6} \);
D \( \frac{3x + 9}{x^2 - 5x + 6} \);

The graph of \( f(x) = 3(x - 1)^2 - 3 \) is shifted up 2 units. Which graph represents the transformation?
35 Randy shades several squares in his grid paper as shown below.

What is the probability that a randomly selected point on the paper lies in a shaded square?

- A 0.15
- B 0.4
- C 0.6
- D 0.75

36 Max dropped an object from a window 100 feet above ground level. The equation \(-16t^2 + 100 = 0\) can be used to determine the time in seconds it will take the object to hit the ground. About how long will the object be in the air before it hits the ground?

- A 2.5 seconds
- B 4 seconds
- C 6.25 seconds
- D 9.2 seconds

37 If \(x = 12\) in the right triangle below, then what is the value of \(y\)?

\[
\begin{align*}
\text{A} & \quad 4 \\
\text{B} & \quad 4\sqrt{3} \\
\text{C} & \quad 6\sqrt{2} \\
\text{D} & \quad 24
\end{align*}
\]

38 Which of the following phrases best describes the translation of the graph of \(f(x) = (x + 2)^2 + 1\) to the graph of \(f(x) = (x - 2)^2 + 2\) in the coordinate plane?

- A a shift 1 unit up and a shift 4 units right
- B a shift 3 units right
- C a shift 1 unit up and a shift 4 units left
- D a shift 4 units up and a shift 1 unit right
39 Yvonne is calculating the total area \( A \) for a garden with a walkway. The walkway will surround the garden on all sides. The garden will be 5 meters in width and \( n \) meters in length. The walkway will be 1 meter in width. The table below shows the relationship between the length of the garden \( n \) and the total area \( A \), including garden and walkway.

<table>
<thead>
<tr>
<th>Length, ( n ) (m)</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area, ( A ) (m²)</td>
<td>84</td>
<td>119</td>
<td>154</td>
<td>189</td>
<td>224</td>
</tr>
</tbody>
</table>

Which function best describes the relationship between length \( n \) and total area \( A \)?

- **A** \( A = 5n + 34 \)
- **B** \( A = 6n + 24 \)
- **C** \( A = 7n + 14 \)
- **D** \( A = n^2 - 16 \)

41 Simplify the equation below. What restrictions must be placed on \( x \)?

\[
y = \frac{2x^2 + 6x}{x^2 - 9}
\]

- **A** \( y = \frac{2x}{x + 3} \);
  \( x \neq -3 \) and \( x \neq 3 \)
- **B** \( y = \frac{2x}{x + 3} \);
  \( x \neq -3, x \neq 0, \) and \( x \neq 3 \)
- **C** \( y = \frac{2x}{x - 3} \);
  \( x \neq -3 \) and \( x \neq 3 \)
- **D** \( y = \frac{2x}{x - 3} \);
  \( x \neq -3, x \neq 0, \) and \( x \neq 3 \)

42 Which of the following is sufficient to guarantee congruence of quadrilaterals?

- **A** AAAS
- **B** SSSS
- **C** SASA
- **D** SASAS

---

Go on
43 If \( g(x) = x^2 - 4x + 7 \) and 
\( h(x) = 5x^2 + x - 1 \),
what is \( g(x) + h(x) \)?

\[\begin{align*}
\text{A} \quad & (g + h)(x) = 5x^2 - 3x + 6 \\
\text{B} \quad & (g + h)(x) = 6x^2 - 5x + 6 \\
\text{C} \quad & (g + h)(x) = 6x^2 - 3x + 6 \\
\text{D} \quad & (g + h)(x) = 6x^2 - 3x + 8 
\end{align*}\]

44 Which statement was used to create the truth table below?

<table>
<thead>
<tr>
<th>4 math credits</th>
<th>4 English credits</th>
<th>Requirements met?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

\[\begin{align*}
\text{A} \quad & (x - 3)^2 + (y - 2)^2 = 16 \\
\text{B} \quad & (x - 3)^2 + (y + 2)^2 = 16 \\
\text{C} \quad & (x - 3)^2 + (y - 2)^2 = 4 \\
\text{D} \quad & (x + 3)^2 + (y - 2)^2 = 16 
\end{align*}\]

45 A circle is graphed on the coordinate plane as shown below.

What is the equation of a congruent circle that is a reflection across the y-axis?

\[\begin{align*}
\text{A} \quad & 2M + \begin{bmatrix} -3 & -3 & -3 \\ 5 & 5 & 5 \end{bmatrix} \\
\text{B} \quad & M + M \\
\text{C} \quad & M - \begin{bmatrix} 2 & 2 & 2 \\ -5 & -5 & -5 \end{bmatrix} \\
\text{D} \quad & M + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} 
\end{align*}\]

46 The vertices for triangle \( RST \) are 
\( R(2, -3) \), \( S(3, 4) \), and \( T(-3, 1) \). If \( RST \) is represented by the vertex matrix \( M \),
which of the following transformations will result in a vertex matrix for a triangle that is congruent to \( RST \)?

\[\begin{align*}
\text{A} \quad & 2M + \begin{bmatrix} -3 & -3 & -3 \\ 5 & 5 & 5 \end{bmatrix} \\
\text{B} \quad & M + M \\
\text{C} \quad & M - \begin{bmatrix} 2 & 2 & 2 \\ -5 & -5 & -5 \end{bmatrix} \\
\text{D} \quad & M + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} 
\end{align*}\]
47 In the figure shown below, which given information would be sufficient to prove that \( \triangle XYZ \cong \triangle WZY \) by ASA?

\[ \begin{align*}
X & \quad Y \quad W \\
\quad Z
\end{align*} \]

A \( XY \parallel Zw \) and \( XZ \parallel YW \)
B \( YW \cong XZ \) and \( XY \cong WZ \)
C \( YW \cong XZ \) and \( XZ \parallel YW \)
D \( \angle X \cong \angle W \) and \( XY \parallel ZW \)

48 Which expression is equivalent to \( \frac{x + 1}{x^2 + 5x} \div \frac{x}{x^2 + 3x - 10} \)?

A \( \frac{x + 1}{x(x + 5)} \cdot \frac{1}{(x - 2)(x + 5)} \)
B \( \frac{x + 1}{x^2} \cdot \frac{x - 2}{x} \)
C \( \frac{x + 1}{x} \cdot \frac{x + 2}{x} \)
D \( \frac{x + 1}{x} \cdot \frac{x - 2}{x} \)

49 Given: Polygon \( EFGHJ \sim KLMNP \).

If \( LM = 27 \text{ feet} \), what is the length of \( MN \)?

A \( \frac{4}{3} \text{ ft} \)
B \( \frac{16}{3} \text{ ft} \)
C \( 12 \text{ ft} \)
D \( 24 \text{ ft} \)

50 Subtract

\[ \begin{bmatrix}
6 & -3 & -7 \\
8 & -10 & -8
\end{bmatrix}
\] \[ \begin{bmatrix}
-2 & -12 & -3 & -7 \\
-4 & 5 & -8
\end{bmatrix} \]

A \[ \begin{bmatrix}
-18 & 3 & 7 \\
16 & 0 & 8
\end{bmatrix} \]
B \[ \begin{bmatrix}
30 & -9 & -21 \\
0 & -20 & -24
\end{bmatrix} \]
C \[ \begin{bmatrix}
-18 & -9 & -21 \\
0 & 0 & -24
\end{bmatrix} \]
D \[ \begin{bmatrix}
30 & 3 & 7 \\
16 & -20 & 8
\end{bmatrix} \]
Diagnostic Test  (continued)

51 Which of the following expressions is equivalent to \( \frac{3}{(27x)^{\frac{1}{2}}} \)?

A \( \sqrt{\frac{3x}{3x^2}} \)

B \( \sqrt{\frac{3x}{3x|x|}} \)

C \( \frac{1}{x|x|} \)

D \( \sqrt{\frac{3x}{3x}} \)

52 Quadrilateral EFGH below is a parallelogram.

![Parallelogram Diagram]

Which argument proves that EFGH is also a square?

A Since all the sides are congruent, EFGH is a square.

B Since \( \overline{EG} \cong \overline{FH} \), EFGH is a square.

C Since the diagonals are congruent and \( \overline{EF} \cong \overline{HG} \), EFGH is a square.

D Since the diagonals are congruent and \( \overline{EF} \cong \overline{FG} \), EFGH is a square.

53 The graph of \( f(x) = -\frac{1}{2}(x - 3)^2 + 2 \) is shown below.

![Graph Diagram]

Use the graph. What are all the zeros of the function?

A 1 and 5

B 1, 3, and 5

C -2.5, 1, and 5

D -2.5, 1, 3, and 5

54 There are 4 yellow stickers and 3 blue stickers in a bag. Evan pulls out a yellow sticker and keeps it. Then Karen pulls out a blue sticker. In which situation would the chances be the greatest for the next student after Karen to pick a blue sticker?

A Karen keeps the blue sticker.

B Karen returns the blue sticker to the bag.

C Karen returns the blue sticker to the bag. Then she pulls out a different blue sticker and keeps it.

D Karen returns the blue sticker to the bag. Then she pulls out a yellow sticker and keeps it.
55 A 120-foot bridge support column is stabilized by four support cables using a total of 1290 feet of cable. Approximately how far from the base of the column is each support cable planted? Assume that all cables are the same length.

A 299.3 ft  
B 344.1 ft  
C 1197.4 ft  
D 1284.4 ft

56 Simplify \( \log_6 6x^m \).

A \( m \log_6 x \)  
B \( 1 + m \log_6 x \)  
C \( 6 + m \log_6 x \)  
D \( m + m \log_6 x \)

57 Which are logically equivalent?

A conditional and inverse  
B contrapositive and conditional  
C converse and inverse  
D inverse and contrapositive

58 Researchers at Duke University in Durham, North Carolina, are developing techniques to make objects invisible. The techniques involve refraction, which is the bending of light as it moves through transparent materials such as water or glass. The index of refraction is the ratio of the speed of light through a material and the speed of light through a vacuum. The index is given by the function \( f(x) = \frac{x}{3 \times 10^8} \), where \( x \) is the speed of light through the material and \( 3 \times 10^8 \) is the speed of light through the vacuum, both are measured in meters per second. If there is another function, \( g \), such that, \( g(f(x)) = x \), what is \( g \)?

A \( g(x) = (3 \times 10^9)(x) \)  
B \( g(x) = \frac{3 \times 10^8}{x} \)  
C \( g(x) = \frac{1}{(3 \times 10^8)(x)} \)  
D \( g(x) = -\frac{x}{3 \times 10^8} \)
59 Emily draws a point and a line on a piece of paper. She folds the paper at the point so that her original line coincides with itself as shown in the diagram below.

Which statement represents the geometric relationship illustrated by this fold?

A If consecutive interior angles are supplementary, then lines are parallel.
B If alternate interior angles are congruent, then lines are parallel.
C Given a line and a point not on that line, there is exactly one line perpendicular to the given line which contains the given point.
D There are an infinite number of perpendicular lines to a given line.

60 Which graph represents the function $f(x) = 2x^2 - 3$?


Diagnostic Test (continued)

61 A store sells red, blue, black, orange, and white T-shirts in small, medium, and large sizes. The information is organized in a matrix, with each element representing the number of T-shirts sold for a different color and size combination. Which of the following statements best describes the matrix?

A. The matrix has 8 elements.
B. The matrix has 5 rows and 3 columns.
C. The matrix has 3 more rows than columns.
D. The matrix has 15 columns and 1 row.

62 In 1979, one of the world’s largest windmills was built on top of Howard’s Knob, a mountain in Boone, North Carolina. Although the experimental windmill was designed to power 300 to 500 average-sized homes, it had to be shut down in 1983 because it was too loud. The windmill consisted of two blades, each of which measured 30 meters from the center of the windmill to the tip. If the tip of one blade rotated 30° counterclockwise from a straight upward position, what was the approximate vertical distance from the initial position of the blade to its finishing position?

A. 4.0 feet
B. 8.8 feet
C. 15.0 feet
D. 26.0 feet

63 Lakisha is making a bin out of a flat rectangular piece of cardboard. The cardboard is 12 inches by 8 inches. She cuts four squares of length x, in inches, out from the corners. Then she folds the cardboard up along the dotted lines to make the sides of the bin.

What is the value of x if the base of the bin has an area of 32 square inches?

A. 1
B. 2
C. 4
D. 8

64 Max dropped an object from a window 100 feet above ground level. The function \( f(t) = -16t^2 + 100 \) can be used to determine the height of the object after t seconds. Approximately how long will the object be in the air before it hits the ground?

A. 0.4 second
B. 1.25 seconds
C. 2.5 seconds
D. 6.25 seconds
Rosa wants to organize a bake sale to raise money for her environmental care club. She makes a list of all the necessary tasks, approximately how many days each task will take to complete, and what tasks must precede other tasks.

<table>
<thead>
<tr>
<th>Task Code</th>
<th>Task</th>
<th>Time (days)</th>
<th>Prior Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Pick recipes and date of bake sale</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Assign recipes to volunteer bakers</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>Rent tables</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>Design posters and flyers</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>Print posters and flyers</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>Advertise</td>
<td>14</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>Collect finished baked goods and wrap in individual serving sizes</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>Set up table and lay out baked goods</td>
<td>1</td>
<td>6,7</td>
</tr>
</tbody>
</table>

Rosa has enough help that many tasks can be in progress at one time. What is the minimum number of days needed for Rosa's bake sale?

- A 17 days
- B 20 days
- C 24 days
- D 34 days

Pilar creates the graph below to display the least-squares regression line for the eight data points shown.

She calculates the slope $m$ of the least-squares regression line and the correlation coefficient $r$ for the data. Which of the following statements best describes how $m$ and $r$ would change if the point with coordinates (40, 49) is added to the data?

- A Both $m$ and $r$ get closer to 0.
- B Both $m$ and $r$ get closer to $-1$.
- C While $m$ gets closer to 0, $r$ gets closer to $-1$.
- D While $m$ gets closer to $-1$, $r$ gets closer to 0.

What is the solution to the equation $4^x = 12$?

- A $x = \frac{\log 12}{\log 4}$
- B $x = \frac{\log 4}{\log 12}$
- C $x = 3$
- D $x = \frac{1}{3}$
68 Given:
\[ \angle WXZ \text{ and } \angle ZXY \text{ form a linear pair.} \]
Using the given statement, which conclusion can be inferred?

- A. \( \angle WXZ \equiv \angle ZXY \)
- B. \( m\angle WXZ + m\angle ZXY = 90 \)
- C. Points W, X, and Y are collinear.
- D. \( \angle WXZ \) and \( \angle WXY \) are supplementary.

69 What is the solution to the system of equations shown below?
\[
\begin{align*}
2x - 3y + 4z &= 3 \\
x + y + z &= 6 \\
4x - 8y + 4z &= 12
\end{align*}
\]
- A. \( (0, 4, 2) \)
- B. \( (7, 1, -2) \)
- C. no solutions
- D. infinitely many solutions

70 Which function is the inverse of \( f(x) = \frac{1}{2}x + 4 \)?
- A. \( f^{-1}(x) = -\frac{1}{2}x - 4 \)
- B. \( f^{-1}(x) = 2x - 8 \)
- C. \( f^{-1}(x) = 2x - 4 \)
- D. \( f^{-1}(x) = 2x + 8 \)

71 Laura plays softball, soccer, and basketball. She has one game every Saturday at the possible fields listed below.

- Softball
  - Prospect Field
  - Drew Field

- Soccer
  - Arlington Field
  - Reading Field
  - Mason Field

- Basketball
  - Locks Field
  - Prospect Field

If all options are equally likely, what is the probability that she will have a game at Mason Field on a random Saturday?

- A. \( \frac{2}{3} \)
- B. \( \frac{2}{9} \)
- C. \( \frac{1}{6} \)
- D. \( \frac{1}{9} \)

72 Which function has a graph for which \( f(x) \to \infty \) as \( x \to -\infty \) and as \( x \to \infty \)?
- A. \( f(x) = x^5 - x^4 + x^3 \)
- B. \( f(x) = x^6 - x^5 + x^4 \)
- C. \( f(x) = -x^5 - x^4 + x^3 \)
- D. \( f(x) = -x^6 - x^5 + x^4 \)
Which graph represents the system of inequalities below?

\[ 2x - y + 2 < 0 \]
\[ x + 2 \leq 0 \]

A)  

B)  

C)  

D)  

What are the \( x \)- and \( y \)-intercepts of the graph of \( g(x) = x^3 + 8 \)?

A) \( x \)-intercept = (0, 8)  
   \( y \)-intercept = (2, 0)  

B) \( x \)-intercept = (2, 0)  
   \( y \)-intercept = (0, 8)  

C) \( x \)-intercept = (−2, 0)  
   \( y \)-intercept = (0, 8)  

D) \( x \)-intercept = (−2, 0)  
   \( y \)-intercept = (0, −8)

In Greenville, North Carolina, the Ground Cloud is a 12-foot circular fountain illuminated at night. When Maria looks at the fountain, only the portion shown below is illuminated.

If the edge of the lighted portion of the fountain is 4 feet long, what is the arc measure of the lighted portion?

A) 9.4°  
B) 12.7°  
C) 19.1°  
D) 38.2°

Go on
North Carolina is the largest producer of sweet potatoes in the United States. The table below shows sweet potato production for the years 2000 to 2008.

<table>
<thead>
<tr>
<th>Years since 2000</th>
<th>Production (millions of lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>555</td>
</tr>
<tr>
<td>1</td>
<td>558</td>
</tr>
<tr>
<td>2</td>
<td>481</td>
</tr>
<tr>
<td>3</td>
<td>588</td>
</tr>
<tr>
<td>4</td>
<td>688</td>
</tr>
<tr>
<td>5</td>
<td>595</td>
</tr>
<tr>
<td>6</td>
<td>702</td>
</tr>
<tr>
<td>7</td>
<td>666.5</td>
</tr>
<tr>
<td>8</td>
<td>874</td>
</tr>
</tbody>
</table>

Using the least-squares regression line for the data, predict the approximate production of sweet potatoes in North Carolina for the year 2020.

A 684 million pounds  
B 839 million pounds  
C 1009 million pounds  
D 1181 million pounds

Which table of values best represents a radical function?

A | x | f(x) |
--|---|------|
| 1 | -1 |
| 4 | -4 |
| 9 | -9 |
| 16| -16|
| 25| -25|

B | x | f(x) |
--|---|------|
| 1 | 1  |
| 4 | 2  |
| 9 | 3  |
| 16| 4  |
| 25| 5  |

C | x | f(x) |
--|---|------|
| 1 | 2  |
| 4 | 8  |
| 9 | 18 |
| 16| 32 |
| 25| 50 |

D | x | f(x) |
--|---|------|
| 1 | 1  |
| 4 | 16 |
| 9 | 81 |
| 16| 256|
| 25| 625|
Diagnostic Test (continued)

78 A pizza delivery driver needs to deliver a pizza to Gary’s house. Possible routes with the travel times in minutes are shown in the vertex-edge graph below.

What is the earliest time Gary can expect his pizza if the driver leaves the pizza shop at 7:10 PM?

A 7:11 PM  
B 7:21 PM  
C 7:23 PM  
D 7:31 PM

79 What are all the values of x for which \( f(x) = \frac{x^2 - 5x + 4}{x^2 + 3x - 4} \) is discontinuous?

A -4, 1, 4  
B 1, 4  
C -4, 1  
D -4

80 Given: Triangle \( LMN \) is an isosceles triangle with vertex angle \( M \).
Prove: Angles 1 and 2 are congruent.

Which statement would be made in an indirect proof?

A Triangle \( LMN \) is not an isosceles triangle.  
B Angle \( M \) is not a vertex angle.  
C Angles 1 and 2 are not congruent.  
D Points \( L, M, \) and \( N \) do not form a triangle.
**Practice By Standard**

**Clarifying Objective MBC.A.4.1**

<table>
<thead>
<tr>
<th>1</th>
<th>Add ( \frac{x + 1}{x - 2} + \frac{5}{x^2 + x - 6} ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( \frac{x + 8}{x^2 + x - 6}; x \neq 2 ) and ( x \neq -3 )</td>
</tr>
<tr>
<td>B</td>
<td>( \frac{x^2 + 9x + 6}{x^2 + x - 6}; x \neq 2 ) and ( x \neq -3 )</td>
</tr>
<tr>
<td>C</td>
<td>( \frac{x + 6}{x - 2}; x \neq 2 ) and ( x \neq -3 )</td>
</tr>
<tr>
<td>D</td>
<td>( \frac{x^2 + 4x + 8}{x^2 + x - 6}; x \neq 2 ) and ( x \neq -3 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>2</th>
<th>Subtract ( \frac{1}{x^2} - \frac{x + 3}{2x} ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( \frac{-x^2 - 3x + 2}{2x^2}; x \neq 0 )</td>
</tr>
<tr>
<td>B</td>
<td>( \frac{x + 2}{2x^2}; x \neq 0 )</td>
</tr>
<tr>
<td>C</td>
<td>( \frac{-x^2 + 3x + 2}{2x^2}; x \neq 0 )</td>
</tr>
<tr>
<td>D</td>
<td>( \frac{1 - 3x^2}{2x^2}; x \neq 0 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>3</th>
<th>Divide ( \frac{x^2 - 4}{x^2 - x - 2} ÷ \frac{1}{3x + 3} ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( x + 2; x \neq -1 ) and ( x \neq 2 )</td>
</tr>
<tr>
<td>B</td>
<td>( x - 2; x \neq -1 ) and ( x \neq 2 )</td>
</tr>
<tr>
<td>C</td>
<td>( 3x + 6; x \neq -1 ) and ( x \neq 2 )</td>
</tr>
<tr>
<td>D</td>
<td>( 3x - 6; x \neq -1 ) and ( x \neq 2 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>4</th>
<th>Janelle and Rick are both riding in a 110-mile bike race from Charlotte, North Carolina, to Chapel Hill, North Carolina. Janelle completes the course in ( x ) hours and Rick completes the course in ( x + 2 ) hours. The expression ( \frac{110}{x} - \frac{110}{x + 2} ) models the difference between their average speeds. Which expression is equivalent to ( \frac{110}{x} - \frac{110}{x + 2} )?</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( \frac{110}{(x + 2)}; x \neq 0 ) and ( x \neq -2 )</td>
</tr>
<tr>
<td>B</td>
<td>( \frac{110}{(x)(x + 2)}; x \neq 0 ) and ( x \neq -2 )</td>
</tr>
<tr>
<td>C</td>
<td>( \frac{220}{(x + 2)}; x \neq 0 ) and ( x \neq -2 )</td>
</tr>
<tr>
<td>D</td>
<td>( \frac{220}{(x)(x + 2)}; x \neq 0 ) and ( x \neq -2 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>5</th>
<th>Multiply ( \frac{x^2 - 3x}{x + 2} · \frac{x^2 - 4}{x^3 - 5x^2 + 6x}. )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( \frac{1}{x + 2}; x \neq 0, x \neq 2, x \neq -2, ) and ( x \neq 3 )</td>
</tr>
<tr>
<td>B</td>
<td>( \frac{x}{x + 2}; x \neq 2, x \neq -2, ) and ( x \neq 3 )</td>
</tr>
<tr>
<td>C</td>
<td>( \frac{x + 2}{x + 2}; x \neq 2, ) and ( x \neq -2 )</td>
</tr>
<tr>
<td>D</td>
<td>( \frac{x - 3}{x + 2}; x \neq 2, ) and ( x \neq -2 )</td>
</tr>
</tbody>
</table>
Practice By Standard
Clarifying Objective MBC.A.4.2

1 Celia is explaining how to simplify the rational expression \( \frac{x^2 - 3x - 4}{x^2 - 2x - 8} \). She says that the first step is to factor both the numerator and the denominator. Which expression is equivalent to \( \frac{x^2 - 3x - 4}{x^2 - 2x - 8} \)?

A \( \frac{(x + 1)(x + 4)}{(x + 4)(x + 2)} \)
B \( \frac{(x - 1)(x + 4)}{(x + 4)(x - 2)} \)
C \( \frac{(x + 1)(x - 4)}{(x - 4)(x + 2)} \)
D \( \frac{(x - 1)(x - 4)}{(x - 4)(x - 2)} \)

2 What is the simplest form of \( \frac{1}{1 + \frac{1}{x - 2}} \)?

A \( \frac{x - 2}{x - 1} \); \( x \neq 1 \) and \( x \neq 2 \)
B \( \frac{x - 2}{2} \); \( x \neq 1 \) and \( x \neq 2 \)
C \( \frac{1}{x - 1} \); \( x \neq 1 \) and \( x \neq 2 \)
D \( \frac{x - 2}{x - 3} \); \( x \neq 1 \) and \( x \neq 2 \)

3 Which expression is the simplest form of \( \frac{x - 3}{x + 2} \div \frac{x - 3}{x^2 - 4} \)?

A \( \frac{(x - 3)(x^2 - 4)}{(x + 2)(x - 3)} \)
B \( \frac{(x - 3)(x^2 - 4)}{x \neq -2, x \neq 2, \text{ and } x \neq 3} \)
C \( \frac{(x - 3)(x + 2)}{x \neq -2, x \neq 2, \text{ and } x \neq 3} \)
D \( x - 2 \)

4 Which expression is equivalent to \( \frac{x^2 - 2x}{x^2 - 4x - 5} \)?

A \( \frac{x^2 - 2x}{x^2 + 4x - 5} \frac{3x}{2x - 10} \)
B \( \frac{x - 2}{x + 1} \frac{2}{3} \)
C \( \frac{1}{x - 3} \frac{2x - 10}{3} \)
D \( \frac{1}{x + 1} \frac{1}{3x} \)

x \neq -1, x \neq 0, \text{ and } x \neq 5
Practice By Standard
Clarifying Objective MBC.A.5.1

1. If \( f(x) = 4x^2 - 5x + 2 \) and \( g(x) = x^3 + 2x \), what is \( f(x) - g(x) \)?
   - A. \( f(x) - g(x) = 3x^2 - 7x + 2 \)
   - B. \( f(x) - g(x) = -x^3 + 4x^2 - 7x + 2 \)
   - C. \( f(x) - g(x) = x^3 + 4x^2 - 7x + 2 \)
   - D. \( f(x) - g(x) = -x^3 + 4x^2 - 3x + 2 \)

2. If \( g(x) = 3x - 4 \) and \( h(x) = x^2 - 1 \), what is \( g(x) + h(x) \)?
   - A. \( g(x) + h(x) = 3x^3 + 4 \)
   - B. \( g(x) + h(x) = 4x^2 - 5 \)
   - C. \( g(x) + h(x) = -x^2 - 3x + 5 \)
   - D. \( g(x) + h(x) = x^2 + 3x - 5 \)

3. A cell phone service charges access fees and late fees. Both fees are percentages of the basic monthly service cost. Suppose \( x \) represents the cost of basic monthly service, \( f(x) = 0.03x \) represents the access fee, and \( g(x) = 0.05x \) represents the late fee. Which function can be used for the total of both fees, \( f + g \)?
   - A. \( (f + g)(x) = 0.08x \)
   - B. \( (f + g)(x) = 1.08x \)
   - C. \( (f + g)(x) = x + 0.08 \)
   - D. \( (f + g)(x) = x + 1.08 \)

4. What is the product of the functions \( f(x) = \frac{x + 3}{x + 2} \) and \( g(x) = \frac{x^2 - 9}{x^2 + 5x + 6} \)?
   - A. \( (f \cdot g)(x) = \frac{x + 3}{x - 3}; x \neq 3 \)
   - B. \( (f \cdot g)(x) = \frac{x - 3}{x + 3}; x \neq -3 \)
   - C. \( (f \cdot g)(x) = \frac{x^2 - 9}{x^2 + 4x + 4}; x \neq -2 \) and \( x \neq -3 \)
   - D. \( (f \cdot g)(x) = \frac{x^2 + 6x + 9}{x^2 - 4}; x \neq -2 \) and \( x \neq 2 \)

5. If \( h(x) = -4x \) and \( j(x) = 6x \), what is \( h(x) \div j(x) \)?
   - A. \( \frac{2}{3} \)
   - B. \( -\frac{2}{3} \)
   - C. \( \frac{2}{3}x \)
   - D. \( -\frac{2}{3}x \)

6. Which function represents \( f(x) \cdot g(x) + f(x) \) if \( f(x) = x + 3 \) and \( g(x) = x - 5 \)?
   - A. \( f(x) \cdot g(x) + f(x) = 2x^2 + 4x - 6 \)
   - B. \( f(x) \cdot g(x) + f(x) = x^2 - 2x - 15 \)
   - C. \( f(x) \cdot g(x) + f(x) = x^2 - x - 12 \)
   - D. \( f(x) \cdot g(x) + f(x) = 3x + 1 \)
Practice By Standard
Clarifying Objective MBC.A.5.2

1 Which function is the inverse of \( f(x) = 3x - 1 \)?
   - A \( f^{-1}(x) = 3x + 1 \)
   - B \( f^{-1}(x) = -3x - 1 \)
   - C \( f^{-1}(x) = \frac{x - 1}{3} \)
   - D \( f^{-1}(x) = \frac{x + 1}{3} \)

2 Which functions are inverses of each other?
   - A \( f(x) = \frac{2}{3}x - 1 \)
     \( g(x) = \frac{3}{2}x + \frac{3}{2} \)
   - B \( f(x) = \frac{2}{3}x - 1 \)
     \( g(x) = -\frac{2}{3}x + 1 \)
   - C \( f(x) = \frac{3}{2}x + 3 \)
     \( g(x) = -\frac{3}{2}x - \frac{3}{2} \)
   - D \( f(x) = \frac{3}{2}x + 3 \)
     \( g(x) = -\frac{2}{3}x - 1 \)

3 The graph of a function is shown below.

Which function could be the inverse of the function in the graph?
   - A \( y = \frac{1}{2}x + 2 \)
   - B \( y = -\frac{1}{2}x + 4 \)
   - C \( y = 2x + 2 \)
   - D \( y = -2x + 4 \)

4 Which table represents a function that has an inverse that is also a function?

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>f(x)</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>x</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>f(x)</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>x</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>f(x)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>x</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>f(x)</td>
<td>6</td>
<td>2</td>
<td>8</td>
<td>6</td>
</tr>
</tbody>
</table>
1. The functions \( g(x) \) and \( g^{-1}(x) \) are inverses of each other. What is the value of \( g^{-1}(g(x)) \)?
   - A. 0
   - B. 1
   - C. \(-1\)
   - D. \( x \)

2. The highest temperature on record in North Carolina is 110°F, recorded in Fayetteville in 1983. The function \( f(x) = \frac{5}{9}(x - 32) \) converts degrees Fahrenheit to degrees Celsius and the function \( f^{-1}(x) = \frac{9}{5}x + 32 \) converts degrees Celsius to degrees Fahrenheit. What is the value of \( f^{-1}(f(110)) \)?
   - A. 43°F
   - B. 78°F
   - C. 110°F
   - D. 144°F

3. If the functions \( h(x) \) and \( h^{-1}(x) \) are inverses of each other, what is the value of \( h(h^{-1}(1)) \)?
   - A. 0
   - B. 1
   - C. \(-1\)
   - D. \( x \)

4. If the functions \( f(x) \) and \( f^{-1}(x) \) are inverses of each other, what generalization can be made about \( f(f^{-1}(x)) \) and \( f^{-1}(f(x)) \)?
   - A. \( f(f^{-1}(x)) = f^{-1}(f(x)) \)
   - B. \( f(f^{-1}(x)) + f^{-1}(f(x)) = 0 \)
   - C. \( f(f^{-1}(x)) = \frac{1}{f^{-1}(f(x))} \)
   - D. \( f^{-1}(f(x)) = \frac{1}{f(f^{-1}(x))} \)

5. Suppose \( f(x) = 4x + 2 \) and \( g(x) = \frac{1}{4}x - \frac{1}{2} \). If \( f(g(x)) = x \), which statement best describes the relationship between the functions \( f \) and \( g \)?
   - A. The functions \( f \) and \( g \) are equivalent.
   - B. The functions \( f \) and \( g \) are inverses of each other.
   - C. For any value of \( x \), \( f(x) + g(x) = 0 \).
   - D. For any value of \( x \), \( f(x) \cdot g(x) = 1 \).
Practice By Standard
Clarifying Objective MBC.A.5.4

1. The table below displays several points of an exponential function.

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>36</td>
</tr>
<tr>
<td>3</td>
<td>216</td>
</tr>
<tr>
<td>4</td>
<td>1296</td>
</tr>
<tr>
<td>5</td>
<td>7776</td>
</tr>
</tbody>
</table>

Which logarithmic function is an inverse of the function in the table?

- **A** $f^{-1}(x) = \log_6 x$
- **B** $f^{-1}(x) = \log_6 x^2$
- **C** $f^{-1}(x) = 6 \log_6 x$
- **D** $f^{-1}(x) = \log_6 x^6$

2. If $f(x) = 3^x$ and $g(x) = \log_3 x$, what is $g(f(x))$?

- **A** 3
- **B** $x$
- **C** $3^x$
- **D** $\log_3 x$

3. What is the inverse of $g(x) = \log_{10} x$?

- **A** $g^{-1}(x) = 10$
- **B** $g^{-1}(x) = 10x$
- **C** $g^{-1}(x) = x^{10}$
- **D** $g^{-1}(x) = 10^x$

4. If $h(x) = \log_8 x$ and $h^{-1}(x) = 8^x$, what is $h^{-1}(h(2))$?

- **A** 2
- **B** 4
- **C** 16
- **D** 64

5. The function $f(x) = 2^x$ is shown in the graph below.

Which statement best describes the relationship between the graph of $f(x) = 2^x$ and the graph of $g(x) = \log_2 x$?

- **A** The graphs are reflections of each other over the x-axis.
- **B** The graphs are reflections of each other over the y-axis.
- **C** The graphs are reflections of each other over the line $y = x$.
- **D** The graphs are reflections of each other over the line $y = -x$.
Practice By Standard
Clarifying Objective MBC.A.6.1

1 Which equation is equivalent to \( \log_2 16 = x \)?
A \( 16^2 = x^2 \)
B \( 16^2 = x \)
C \( 2^x = 16 \)
D \( 2^{16} = x \)

4 Solve \( \log_9 y = \frac{1}{2} \).
A \( y = 18 \)
B \( y = 4 \frac{1}{2} \)
C \( y = 3 \)
D \( y = \frac{1}{512} \)

2 In 2007, plants were dying at a nursery in Greene County, North Carolina. Tests done by the North Carolina Department of Agriculture and Consumer Services showed the problem was due to well water with an abnormally acidic pH of 3.0. The pH of a water sample is given by the formula \( \text{pH} = -\log_{10} [\text{H}^+] \), where \([\text{H}^+]\) is the hydrogen ion concentration in moles per liter. What was the hydrogen ion concentration of the well water at the nursery?
A 0.001 moles per liter
B 0.003 moles per liter
C 0.300 moles per liter
D 0.477 moles per liter

5 Which equation is equivalent to \( \ln x = 10 \)?
A \( \log_{10} x = 1 \)
B \( \log_{10} e = x \)
C \( e^x = 10 \)
D \( e^{10} = x \)

6 Norah wrote an exponential equation that is equivalent to \( \log_5 \frac{1}{25} = x \). Which equation did she write?
A \( \left( \frac{1}{25} \right)^x = 5 \)
B \( 5^x = \frac{1}{25} \)
C \( 5^{\frac{1}{25}} = x \)
D \( 5 = 25^x \)

7 Which equation is equivalent to \( x^3 = 216 \)?
A \( \log_3 x = 216 \)
B \( \log_3 216 = x \)
C \( \log_x 3 = 216 \)
D \( \log_2 216 = 3 \)
Practice By Standard
Clarifying Objective MBC.A.6.2

1. In photography, an f-stop is a number that represents the amount of light exposure. Exposure values are numbers that refer to certain combinations of f-stops and shutter speeds. Exposure value (EV) is given by the equation \( EV = \log_2 \frac{N^2}{t} \), where \( N \) is the f-stop number and \( t \) is the shutter speed in seconds. What is the exposure value when the f-stop is 8.0 and the shutter speed is \( \frac{1}{2} \) second?
   - A 5
   - B 6
   - C 7
   - D 8

2. What is the value of \( x \) if \( \log_4 4^3 = x? \)
   - A 3
   - B 4
   - C 64
   - D 81

3. Simplify \( \log_8 x^2 y^6. \)
   - A \( 54 \log_8 xy \)
   - B \( 15 \log_8 xy \)
   - C \( 15 + \log_8 x + \log_8 y \)
   - D \( 9 \log_8 x + 6 \log_8 y \)

4. Which expression is equivalent to \( \log_5 \frac{3x}{2y}? \)
   - A \( \log_5 3x - \log_5 2y \)
   - B \( \log_5 3x + \log_5 2y \)
   - C \( (\log_5 3x)(\log_5 2y) \)
   - D \( \log_5 (6xy) \)

5. Which expression is equivalent to \( \log_6 \frac{36x}{y^4}? \)
   - A \( 2 \log_6 x - 4 \log_6 y \)
   - B \( 2 + \log_6 x - 4 \log_6 y \)
   - C \( 8 + 4 \log_6 x - 4 \log_6 y \)
   - D \( 8 + 8 \log_6 x - 8 \log_6 y \)

6. What is the value of \( y \) if \( 5^{\log_5 (2y - 1)} = y? \)
   - A 2
   - B 5
   - C 6
   - D 25

7. Sanjay is using the properties of logarithms to simplify \( 9^{\log_3 (3x - 1)}. \) Which equivalent expression can he write?
   - A \( -9 \log(3x - 1) \)
   - B \( 9 \log(3x - 1) \)
   - C \( \frac{9}{\log(3x - 1)} \)
   - D \( \frac{1}{3x - 1} \)
1. The Koury Natatorium, at the University of North Carolina, has a diving platform that is 5 meters high. If a diver jumps off the platform with an upward velocity of 2.25 meters per second, the dive can be modeled by the equation \( h = -5t^2 + 2t + 5 \), where \( h \) is the height above the water and \( t \) is the time in seconds of the dive. Approximately how long will it take for the diver to hit the water?

- **A** 0.8 second
- **B** 1.2 seconds
- **C** 2.4 seconds
- **D** 2.8 seconds

2. What are the solutions to the equation \( x^2 - 7x + 10 = 0 ? \)

- **A** \( x = 2; x = 5 \)
- **B** \( x = 2; x = -5 \)
- **C** \( x = -2; x = 5 \)
- **D** \( x = -2; x = -5 \)

3. If Pilar graphs the quadratic function \( y = 2x^2 + 2x - 12 \), what are the \( x \)-intercepts of her graph?

- **A** 2 and 3
- **B** -2 and 3
- **C** 2 and -3
- **D** -2 and -3

4. Johanna is solving \( 2x^2 - 3x - 5 = 0 \). Which of the following shows Johanna correctly using the quadratic formula?

- **A** \( x = \frac{-3 \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2} \)
- **B** \( x = \frac{-3 \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)} \)
- **C** \( x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2} \)
- **D** \( x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)} \)

5. Catherine is solving \( 15x^2 + 14x - 8 = 0 \). Which of the following shows Catherine correctly factoring the equation?

- **A** \((3x + 4)(5x - 2) = 8\)
- **B** \((3x + 4)(5x - 2) = 0\)
- **C** \((3x + 4)(5x - 2) = 0\)
- **D** \((3x + 4)(5x - 2) = 0\)

6. What are the roots of the equation \( x^2 - 6x = 0 \)?

- **A** \( x = -3; x = 2 \)
- **B** \( x = 0; x = 6 \)
- **C** \( x = 1; x = 6 \)
- **D** \( x = 3; x = 2 \)

**Go on**
Steve rewrites the equation $y = \frac{x^2 - 4}{x + 2}$ as $y = x - 2$. What restrictions must he identify for the variable $x$?

- **A** $x \neq 2$
- **B** $x \neq -2$
- **C** $x \neq 2$ and $x \neq -2$
- **D** There are no restrictions.

Simplify $\frac{3x + 8}{15x + 40}$.

- **A** $\frac{1}{10}; x \neq 0$
- **B** $\frac{1}{10}; x \neq -\frac{8}{3}$
- **C** $\frac{1}{5}; x \neq 0$
- **D** $\frac{1}{5}; x \neq -\frac{8}{3}$

The equation $y = \frac{x^2 + 4x}{x^2 + 3x - 4}$ can also be written as $y = \frac{x}{x - 1}$. What restrictions must be placed on $x$ when the fraction is in its simplest form?

- **A** $x \neq 1$
- **B** $x \neq 1$ and $x \neq -4$
- **C** $x \neq 0, x \neq 1$, and $x \neq -4$
- **D** There are no restrictions.

Simplify the equation below. What restrictions must be placed on $x$?

$$y = \frac{x^2 - 16}{x^2 + 6x + 8}$$

- **A** $y = \frac{x + 4}{x + 2}$
  $$x \neq -4 \text{ and } x \neq 4$$
- **B** $y = \frac{x - 4}{x - 2}$
  $$x \neq -2 \text{ and } x \neq 2$$
- **C** $y = \frac{x - 4}{x + 2}$
  $$x \neq -4 \text{ and } x \neq -2$$
- **D** $y = \frac{x + 4}{x - 2}$
  $$x \neq -4, x \neq -2 \text{ and } x \neq 2$$

The equation $y = \frac{2x^2 - 12x + 10}{2x + 6}$ has a quadratic numerator and a linear denominator. What restrictions must be placed on $x$?

- **A** $x \neq -3$
- **B** $x \neq 1$
- **C** $x \neq 5$
- **D** $x \neq 1$ and $x \neq 5$
Practice By Standard
Clarifying Objective MBC.A.7.3

1 North Carolina’s state bird, the northern cardinal, can live up to 15 years in the wild. The cardinal’s life span is affected by its basal metabolic rate (BMR), which is given by the formula \( BMR = 5.914(m)^{3/4} \), where \( m \) is the body mass in kilograms and BMR is measured in watts. If the northern cardinal’s body mass is 39.9 grams, what is its basal metabolic rate?

A 0.339 watt  
B 0.528 watt  
C 60.206 watts  
D 93.888 watts

2 Solve \( 4 = \log_4 x \).

A \( x = 1 \)  
B \( x = 16 \)  
C \( x = 64 \)  
D \( x = 256 \)

3 If \( 2^{5x-8} = 16^x \), what is the value of \( x \)?

A \( x = \frac{1}{8} \)  
B \( x = 2 \)  
C \( x = \frac{1}{2} \)  
D \( x = 8 \)

4 Solve \( 2^x = 12 \).

A \( x = \log_2 12 \)  
B \( x = \log 12 - \log 2 \)  
C \( x = 6 \)  
D \( x = \frac{\log 3}{\log 2} + 2 \)

5 The graph of \( 3^{2x} \) is shown below.

Use the graph. Which could be the solution to the equation \( 3^{2x} = 729 \)?

A \( x = 1000 \)  
B \( x = 729 \)  
C \( x = 3 \)  
D \( x = 1.5 \)

6 A certain radioactive element decays over time according to the equation \( y = A \left( \frac{1}{2} \right)^{\frac{t}{300}} \), where \( A \) is the number of grams present initially and \( t \) is time in years. If 100 grams were initially present, how many grams will remain after 600 years?

A 0.8 gram  
B 25 grams  
C 50 grams  
D 100 grams

7 Solve \( \ln 6x - \ln x^2 = 1 \).

A \( x = \frac{6}{e} \)  
B \( x = \sqrt{6e} \)  
C \( x = 6 \)  
D \( x = 3 \pm 2\sqrt{2} \)
Practice By Standard
Clarifying Objective MBC.A.8.1

1. What is the domain of the function
   \[ f(x) = \frac{x^2 - 1}{x^3 + 2x - 3} \]?
   - A. all \( x \)
   - B. all \( x \) not equal to 1 or \(-1\)
   - C. all \( x \) not equal to 1 or \(-3\)
   - D. all \( x \) not equal to 1, \(-1\), or \(-3\)

2. The function \( f(x) = -x^4 + 2 \) is shown in the graph below.

   ![Graph of \( f(x) = -x^4 + 2 \)]

   What are the domain and range of \( f(x) = -x^4 + 2 \)?
   - A. Domain = all real numbers
     Range = all real numbers
   - B. Domain = all real numbers
     Range = \( y \leq 2 \)
   - C. Domain = \( x \leq 2 \)
     Range = all real numbers
   - D. Domain = \( y \leq 2 \)
     Range = \( x \leq 2 \)

3. In 2007, civil engineering students at North Carolina State University participated in a contest to create a concrete cube that meets strict weight and strength requirements. The volume of a cube can be determined by the function \( f(x) = x^3 \), where \( x \) is the length of one side. The theoretical domain of \( f(x) = x^3 \) is all real numbers. Which of the following descriptions best represents the practical domain of \( f(x) = x^3 \) in the given situation?
   - A. The practical domain is all real numbers because the theoretical and practical domains are the same.
   - B. The practical domain is all whole numbers because the length of a side is a whole number.
   - C. The practical domain is all real numbers greater than or equal to zero because the length of a side is a nonnegative number.
   - D. The practical domain is all real numbers greater than zero because the length of a side is a positive number.

4. What is the domain of the radical function \( f(x) = \sqrt{-x + 4} \)?
   - A. \( x \geq 4 \)
   - B. \( x \leq 4 \)
   - C. \( x \geq 2 \)
   - D. \( x \leq 2 \)
Practice By Standard
Clarifying Objective MBC.A.8.2

1 Which point is an x-intercept of $h(x) = \frac{2x - 2}{x^2 - 4}$?
   - **A** (-2, 0)
   - **B** (-1, 0)
   - **C** (1, 0)
   - **D** (2, 0)

2 At what coordinate is there a minimum of the quadratic function $f(x) = 2(x + 3)^2 - 5$?
   - **A** (3, 5)
   - **B** (-3, 5)
   - **C** (3, -5)
   - **D** (-3, -5)

3 Which function is graphed on the coordinate grid shown below?
   - **A** $f(x) = -x^2 - 2x + 1$
   - **B** $f(x) = -x^2 + 4x - 3$
   - **C** $f(x) = x^2 - 4x + 3$
   - **D** $f(x) = x^2 + 2x - 1$

4 What is an x-intercept of the graph of the function $f(x) = 3^x + 4$?
   - **A** (0, 3)
   - **B** (0, 4)
   - **C** (0, 9)
   - **D** The graph has no x-intercept.

5 What are the zeros of the function $g(t) = t^2 + 3t + 2$?
   - **A** -1 and -2
   - **B** 1 and 2
   - **C** $\sqrt{2}/3$ and $-\sqrt{2}/3$
   - **D** $\sqrt{3}/2$ and $-\sqrt{3}/2$

6 What is the y-intercept of $y = -2\cos x + 5$?
   - **A** (0, 5)
   - **B** (0, 1)
   - **C** (0, 3)
   - **D** (0, -5)

7 What is the maximum value of $f(x) = -2x^2 + 8x + 1$?
   - **A** 9
   - **B** 8
   - **C** 1
   - **D** 0

Go on
Practice By Standard
Clarifying Objective MBC.A.8.3

1. At which value does the graph of \( f(x) = \frac{x^2 - 4}{x - 2} \) have a point of discontinuity?
   - A \( x = 4 \)
   - B \( x = 2 \)
   - C \( x = 0 \)
   - D \( x = -2 \)

2. What is the vertical asymptote of the graph of \( f(x) = \frac{2x + 1}{x + 1} \)?
   - A \( x = 0 \)
   - B \( x = -\frac{1}{2} \)
   - C \( x = 1 \)
   - D \( x = -1 \)

3. For which values of \( x \) is the function \( f(x) = x^2 + 4x - 25 \) increasing?
   - A \( x < -2 \)
   - B \( x > -2 \)
   - C \(-2 < x < 2 \)
   - D \( 2 \geq x \geq -2 \)

4. What is the horizontal asymptote of the graph of \( f(x) = 3^{x - 2} \)?
   - A \( y = 3 \)
   - B \( y = 2 \)
   - C \( y = 0 \)
   - D \( y = -2 \)

5. Mitch graphs the function \( f(x) = -2x^2 + 8x - 4 \) on a coordinate plane. What are all the values of \( x \) for which the function is decreasing?
   - A \( x > 2 \)
   - B \( x < 2 \)
   - C \( x > 4 \)
   - D \( x < 4 \)

6. Which of the following functions has a graph with a slant asymptote of \( y = x \)?
   - A \( f(x) = x + \frac{1}{x} \)
   - B \( f(x) = \sqrt{x + 1} \)
   - C \( f(x) = x^3 + x \)
   - D \( f(x) = \frac{x^2 - 1}{x + 1} \)
Practice By Standard
Clarifying Objective MBC.A.8.4

1. Which function has a graph that extends down at each end?
   A. \( f(x) = 2x^4 - x^2 + 5 \)
   B. \( f(x) = 4x^5 + 2x^3 - 6x + 1 \)
   C. \( f(x) = -3x^4 - 2x^3 + x - 5 \)
   D. \( f(x) = x^3 - 3x^2 - 3 \)

2. Which term can be used to determine the end behavior of the polynomial function \( f(x) = x^3 + 4x^2 + 3x + 6 \)?
   A. \( x^3 \)
   B. \( 4x^2 \)
   C. \( 3x \)
   D. \( 6 \)

3. Rosa graphs function \( f \) on a coordinate plane. She observes that as the value of \( x \) becomes increasingly large or increasingly small, the value of \( f(x) \) becomes closer and closer to zero. Which of the following functions has this type of end behavior?
   A. \( f(x) = x^2 \)
   B. \( f(x) = x^3 \)
   C. \( f(x) = \sqrt{x} \)
   D. \( f(x) = \frac{1}{x} \)

4. Which of the following descriptions best represents the end behavior of the graph of \( f(x) = -2x^3 + 4 \)?
   A. Both ends extend up.
   B. Both ends extend down.
   C. The end on the left extends up and the end on the right extends down.
   D. The end on the left extends down and the end on the right extends up.

5. Which function has a graph with one end that extends up and one end that extends down?
   A. \( f(x) = -2x^6 - 3x^4 + x^2 \)
   B. \( f(x) = 5x^5 + 4x^4 - 3x^3 \)
   C. \( f(x) = x^2 + 2x + 1 \)
   D. \( f(x) = x^2 - 2x + 1 \)

6. Which of the following descriptions best represents the end behavior of the graph of \( f(x) = \frac{10x + 2}{-5x^2 - 15} \)?
   A. The left end extends down and the right end extends up.
   B. Both ends extend down.
   C. Both ends extend toward a horizontal asymptote described by \( y = -2 \).
   D. Both ends extend toward a horizontal asymptote described by \( y = 0 \).
1. At a height of 220 feet, the Laurel Creek Bridge is the second tallest bridge in North Carolina. An object is dropped from the bridge into the water below. The table represents the object’s height \( h \), in feet, above the water after 1, 2, and 3 seconds.

<table>
<thead>
<tr>
<th>Time ( t ) (seconds)</th>
<th>Height ( h ) (feet above water)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>220</td>
</tr>
<tr>
<td>1</td>
<td>204</td>
</tr>
<tr>
<td>2</td>
<td>156</td>
</tr>
<tr>
<td>3</td>
<td>76</td>
</tr>
</tbody>
</table>

Which function represents the object’s height \( h \) in feet above the water after \( t \) seconds?

A. \( h = 220 - 16t \)
B. \( h = -16t^2 + 220 \)
C. \( h = 220 - 16^t \)
D. \( h = 220 - \frac{1}{16^t} \)

2. Bacteria in a culture are growing as shown in the table below.

<table>
<thead>
<tr>
<th>Bacteria Growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>

Which statement best describes the growth rate of the bacteria?

A. The number of bacteria is increasing at a linear rate.
B. The number of bacteria is increasing at an exponential rate.
C. The number of bacteria is decreasing at a linear rate.
D. The number of bacteria is decreasing at an exponential rate.

3. Which function could be represented by the graph shown below?

A. \( y = 2^x \)
B. \( y = x^2 - 2x + 1 \)
C. \( y = \frac{x^3}{5} \)
D. \( y = \frac{3}{x^2 - 5} \)
Practice By Standard
Clarifying Objective MBC.A.9.1

1. A clock has a circular face with the numerals 1 through 12 positioned at equal intervals around the outer edge. If the minute hand on the clock rotates 630° clockwise, which statement best describes the position of the minute hand before and after the rotation?
   A. The minute hand starts at 6 and ends at 9.
   B. The minute hand starts at 6 and ends at 12.
   C. The minute hand starts at 3 and ends at 9.
   D. The minute hand starts at 3 and ends at 12.

2. The spinner for a carnival game is a wheel with numbers painted along the edge. The wheel is mounted vertically on the wall. Before Jason spins the wheel, the number he chooses is exactly at the bottom of the wheel. After Jason spins the wheel, that same number is exactly at the top of the wheel. Which description best represents the rotation of the wheel when Jason spins it?
   A. 1050° counterclockwise rotation
   B. 1080° clockwise rotation
   C. 1110° clockwise rotation
   D. 1260° counterclockwise rotation

3. Which situation could best be represented by the graph below?
   A. the distance x, in miles, traveled by a person walking y miles per hour
   B. the position of a rock dropped from a height of y meters after x seconds
   C. the air y, in liters, in a person’s lungs after breathing for x seconds
   D. average speed after walking x miles in y hours

4. A duck bobs up and down as it floats on the water, moving from its highest position to its lowest position every 6 seconds. The distance between the duck’s highest and lowest position is 8 inches. If x represents time, in seconds, and the equilibrium point is y = 0, which function could model the movement of the duck?
   A. y = 3x + 4
   B. y = 3x² + 4x
   C. y = 4 cos \left(\frac{\pi x}{3}\right)
   D. y = 3x² – 4x
Practice By Standard
Clarifying Objective MBC.A.9.2

1 Jolene is pacing back and forth across a rug with a length of 3.5 meters and a width of 1.5 meters, as shown below.

3.5 cm

1.5 cm

It takes Jolene a total of 35 seconds to walk across the length of the rug 20 times. If she continues pacing at the same rate, how many times can she walk across the width of the rug in 30 seconds?

- A 10
- B 25
- C 40
- D 45

2 The pendulum shown in the picture is 0.25 meter long.

The equation \( T = 2\sqrt{0.25} \) can be used to find the approximate time \( T \) in seconds for one full swing of the pendulum. About how many full swings will the pendulum make in 10 seconds?

- A 5
- B 10
- C 20
- D 40

3 Which graph best represents average monthly precipitation that fluctuates between 8 and 3 inches over time?
**Practice By Standard**

**Clarifying Objective MBC.A.9.3**

1. Points $P$ and $Q$ lie on the unit circle shown below.

![Unit Circle Diagram]

What is the horizontal distance between $P$ and $Q$?

- **A** $\frac{\sqrt{3}}{2}$
- **B** $1 + \frac{\sqrt{2}}{2}$
- **C** $1 + \frac{\sqrt{3}}{2}$
- **D** 2

2. When Sarah gets into the seat at the bottom of a Ferris wheel at the Cleveland County Fair, she is 5 feet above the ground. If the seats are 30 feet from the center of the wheel, how far above the ground is Sarah after the Ferris wheel rotates 660° counterclockwise?

- **A** 20 feet
- **B** 25 feet
- **C** 35 feet
- **D** 65 feet

3. A carousel at a carnival has horses arranged in a circle with a radius of 10 feet. Jenna gets onto a horse directly to the right of the center at point $H$, and Tyler gets onto a horse directly above the center at point $G$.

When the ride is over, Jenna and Tyler are at a $\frac{\pi}{3}$ counterclockwise rotation from their starting positions. What is the horizontal distance between Tyler and Jenna at the end of the ride?

- **A** 5 ft
- **B** $5 + 5\sqrt{3}$ ft
- **C** 10 ft
- **D** $10\sqrt{3}$ ft

4. A vertical gear in a machine is in the shape of a circle with a radius of 8 inches. A blue dot has been painted on the edge of the gear. Before the machine starts up, the blue dot is in the bottom position, 3 inches above the floor. When the machine starts, the gear rotates until the blue dot is 7 inches above the floor. Which statement best describes the rotation of the gear?

- **A** The gear rotated 750° clockwise.
- **B** The gear rotated 765° clockwise.
- **C** The gear rotated 780° clockwise.
- **D** The gear rotated 790° clockwise.
1. What is the period for the function \( y = \cos \frac{\theta}{2} \)?
   - A. \( \frac{\pi}{2} \)
   - B. \( \pi \)
   - C. 2\( \pi \)
   - D. 4\( \pi \)

2. The musical note A above middle C has a frequency of 440 hertz. A sine function that represents the behavior of the note A is \( y = 0.2 \sin (880\pi t) \). What is the amplitude of the note A modeled by this function?
   - A. 0.1
   - B. 0.2
   - C. 176
   - D. 4400

3. Which periodic function is equivalent to \( 2 \cos x + 3 \)?
   - A. \( y = 2 \cos \left( x - \frac{\pi}{2} \right) + 3 \)
   - B. \( y = 2 \cos \left( x + \frac{\pi}{2} \right) + 3 \)
   - C. \( y = 2 \sin \left( x - \frac{\pi}{2} \right) + 3 \)
   - D. \( y = 2 \sin \left( x + \frac{\pi}{2} \right) + 3 \)

4. What is the amplitude for the function \( y = \frac{1}{2} \sin 2\theta \)?
   - A. \( \frac{1}{4} \)
   - B. \( \frac{1}{2} \)
   - C. 1
   - D. 2

5. The average high monthly temperatures for Raleigh, North Carolina, are given below.

<table>
<thead>
<tr>
<th>Average High Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raleigh, North Carolina</td>
</tr>
<tr>
<td>Month</td>
</tr>
<tr>
<td>-------------</td>
</tr>
<tr>
<td>January</td>
</tr>
<tr>
<td>February</td>
</tr>
<tr>
<td>March</td>
</tr>
<tr>
<td>April</td>
</tr>
<tr>
<td>May</td>
</tr>
<tr>
<td>June</td>
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<td>July</td>
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<tr>
<td>August</td>
</tr>
<tr>
<td>September</td>
</tr>
<tr>
<td>October</td>
</tr>
<tr>
<td>November</td>
</tr>
<tr>
<td>December</td>
</tr>
</tbody>
</table>

   If \( t = 1 \) represents January, which of the following functions best models this data?
   - A. \( f(t) = -19.5 \cos \left( \frac{\pi}{6}t - 0.5 \right) + 68.5 \)
   - B. \( f(t) = -68.5 \cos \left( \frac{\pi}{6}t - 0.5 \right) + 19.5 \)
   - C. \( f(t) = -38 \cos \left( \frac{\pi}{6}t - 0.5 \right) + 68.5 \)
   - D. \( f(t) = -68.5 \cos \left( \frac{\pi}{6}t - 0.5 \right) + 39 \)
Practice By Standard
Clarifying Objective MBC.A.10.1

1. Karen invests $10,000 in a savings plan that guarantees $200 per year for ten years. Juan invests $10,000 in a savings plan that guarantees 2% interest compounded annually. Which description best represents the situation after ten years?

A. Karen has about $200 more than Juan.
B. Juan has about $200 more than Karen.
C. Karen has about $2000 more than Juan.
D. Juan has about $2000 more than Karen.

2. What function is shown in the graph below?

A. \( f(x) = \frac{1}{x} \)
B. \( f(x) = x \)
C. \( f(x) = x^2 \)
D. \( f(x) = x^3 \)

3. Which type of function could be represented by the graph below?

A. periodic
B. exponential
C. logarithmic
D. absolute value

4. Which function has a vertical asymptote of \( x = 0 \)?

A. \( f(x) = x \)
B. \( f(x) = x^2 \)
C. \( f(x) = |x| \)
D. \( f(x) = \frac{1}{x} \)

5. Which function does not have a y-intercept of \(-2\)?

A. \( y = x - 2 \)
B. \( y = x^2 - 2 \)
C. \( y = \frac{1}{x} - 2 \)
D. \( y = |x| - 2 \)

6. Which type of function repeats its values in regular intervals?

A. quadratic
B. periodic
C. radical
D. rational

Go on
1. The graph of \( f(x) = \log x \) is shifted 2 units down and shrunk horizontally by a factor of 4. What is the equation of the new graph?
   - **A** \( f(x) = \log 2(x - 4) \)
   - **B** \( f(x) = \log 2x - 4 \)
   - **C** \( f(x) = \log 4(x - 2) \)
   - **D** \( f(x) = \log 4x - 2 \)

2. Consider \( f(x) = \frac{1}{x} \) and \( g(x) = \frac{1}{x - 3} \). Which translation will transform the graph of \( f(x) \) into the graph of \( g(x) \)?
   - **A** a shift 3 units right
   - **B** a shift 3 units left
   - **C** a shift 3 units up
   - **D** a shift 3 units down

3. If the graph of the function \( f(x) = 2^x \) is shifted 1 unit down and stretched by a factor of 0.5, which function would represent the new graph?
   - **A** \( f(x) = 0.5(1 - 2^x) \)
   - **B** \( f(x) = 2^{0.5x} - 1 \)
   - **C** \( f(x) = (2 - 1)^{0.5x} \)
   - **D** \( f(x) = 0.5(2^x - 1) \)

4. The graph of \( f(x) = x^2 \) is reflected over the x-axis, shifted up 2 units, and shifted left 2 units. Which is the graph of the new parabola?
   - **A**
   - **B**
   - **C**
   - **D**

---

**Practice By Standard**

**Clarifying Objective MBC.A.10.2**

Go on
The vertices of a quadrilateral are represented using a $2 \times 4$ matrix. If matrix multiplication is used to reflect the quadrilateral across the $x$-axis, how does the new matrix compare to the original matrix?

A. The first row is the same. In the second row, the entries are each multiplied by $-1$.
B. The second row is the same. In the first row, the entries are each multiplied by $-1$.
C. Every entry is multiplied by $-1$.
D. The entries in each row are interchanged with the entries in other rows.

The vertices of triangle $RST$ are $R(3, 5)$, $S(2, -4)$, and $T(0, 4)$. A transformation of $RST$ is represented by the matrix equation shown below.

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 5 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 4 \\ -3 & -2 & 0 \end{bmatrix}$$

Which statement best describes the transformation of $RST$?
A. $RST$ is reflected across the $x$-axis.
B. $RST$ is rotated clockwise $90^\circ$ about the origin.
C. $RST$ is rotated clockwise $180^\circ$ about the origin.
D. $RST$ is rotated clockwise $270^\circ$ about the origin.

Consider the transformation matrix shown below.

$$\begin{bmatrix} -3 & -3 & -3 & -3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

For which transformation would addition of the matrix most likely be used?
A. Translate a quadrilateral 1 unit up and 3 units left.
B. Translate a quadrilateral 3 units down and 1 unit left.
C. Translate a pentagon 1 unit up and 3 units left.
D. Translate a pentagon 3 units down and 1 unit left.

Triangle $EFG$ is transformed into triangle $E'F'G'$ as shown below.

The coordinate matrix for $EFG$ is multiplied by matrix $T$ resulting in the coordinate matrix for $E'F'G'$. What is matrix $T$?
A. $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
B. $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
C. $T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
D. $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
1. Which transformation results in an image that is similar, but not congruent?
   - A. 180° rotation about the origin
   - B. reflection across the line \( x = 2 \)
   - C. dilation by a factor of 2
   - D. translation 2 units up and 7 units right

2. The coordinate matrix for a triangle is multiplied by a transformation matrix \( T \). The resulting triangle is congruent to the original. What is matrix \( T \)?
   - A. \( T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)
   - B. \( T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \)
   - C. \( T = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \)
   - D. \( T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \)

3. The elements of the vertex matrix for a polygon are multiplied by 3.5. Which of the following statements best describes the relationship between the pre-image and image?
   - A. The image and pre-image are similar because the transformation is rigid.
   - B. The image and pre-image are congruent because the transformation is rigid.
   - C. The image and pre-image are similar because the transformation is a dilation.
   - D. The image and pre-image are congruent because the transformation is a dilation.

4. Pre-image matrix \( P \) and image matrix \( I \) for a quadrilateral are shown below.
   \[
   P = \begin{bmatrix} 0 & 3 & -1 & 0 \\ -4 & 1 & 3 & 2 \end{bmatrix}
   \]
   \[
   I = \begin{bmatrix} 0 & 6 & -2 & 0 \\ -8 & 2 & 6 & 4 \end{bmatrix}
   \]
   Which statement is true?
   - A. The transformation is a translation.
   - B. The image is similar and congruent to the pre-image.
   - C. The image is congruent to the pre-image.
   - D. The image is similar but not congruent to the pre-image.

5. A sketch of North Carolina is placed on a coordinate grid as shown.
   If the map is represented as a vertex matrix, which statement best describes the result of multiplying the entries of the matrix by 2?
   - A. The size of the map is halved.
   - B. The size of the map is doubled.
   - C. The map is translated 2 units to the right.
   - D. The map is translated 2 units to the left.
Practice By Standard
Clarifying Objective MBC.G.5.1

1. What is the equation of the graph of a circle with radius 4 and center \((-1, 5)\)?
   - A. \((x - 1)^2 + (y + 5)^2 = 16\)
   - B. \((x + 1)^2 + (y - 5)^2 = 4\)
   - C. \((x + 1)^2 + (y - 5)^2 = 16\)
   - D. \((x - 1)^2 + (y + 5)^2 = 4\)

2. The diameter of a circle has endpoints at \((5, 6)\) and \((-3, 12)\). What is the equation of the circle?
   - A. \((x + 1)^2 + (y - 9)^2 = 25\)
   - B. \((x + 1)^2 + (y + 9)^2 = 25\)
   - C. \((x - 1)^2 + (y - 9)^2 = 25\)
   - D. \((x - 1)^2 + (y + 9)^2 = 25\)

3. The distance by air from Kings Mountain, North Carolina, to Tarboro, North Carolina, is 351.68 miles. Suppose each city is represented as a point on the coordinate plane, with Kings Mountain at the origin and Tarboro at \((351.68, 0)\). If these points are endpoints of the diameter of a circle, what is the equation of the circle?
   - A. \(x^2 + (y - 175.84)^2 = 175.84^2\)
   - B. \(x^2 + y^2 = 175.84^2\)
   - C. \((x - 175.84)^2 + y^2 = 351.68^2\)
   - D. \((x - 175.84)^2 + y^2 = 175.84^2\)

4. A circle is graphed on the coordinate plane as shown below.

   What is the equation of the circle when it is reflected across the x-axis?
   - A. \((x - 3)^2 + (y + 4)^2 = 9\)
   - B. \((x + 3)^2 + (y - 4)^2 = 9\)
   - C. \((x - 3)^2 + (y - 4)^2 = 9\)
   - D. \((x + 4)^2 + (y + 3)^2 = 9\)

5. Circle \(E\) is represented by the equation \((x - 3)^2 + (y - 1)^2 = 16\), and circle \(F\) is represented by the equation \((x - 2)^2 + (y + 1)^2 = 4\). Which of the following statements best describes the relationship between circle \(E\) and circle \(F\)?
   - A. Circle \(E\) intersects circle \(F\) exactly once.
   - B. Circle \(E\) intersects circle \(F\) twice.
   - C. Circle \(E\) is completely inside circle \(F\).
   - D. Circle \(F\) is completely inside circle \(E\).
Practice By Standard
Clarifying Objective MBC.S.2.1

1 Population estimates for Union County, North Carolina, are shown in the table below.

<table>
<thead>
<tr>
<th>Years since 1970</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>54,714</td>
</tr>
<tr>
<td>5</td>
<td>63,000</td>
</tr>
<tr>
<td>10</td>
<td>70,436</td>
</tr>
<tr>
<td>15</td>
<td>76,712</td>
</tr>
<tr>
<td>20</td>
<td>84,210</td>
</tr>
<tr>
<td>25</td>
<td>100,437</td>
</tr>
</tbody>
</table>

A least-squares regression line is used to model the data in the table. Which statement best compares the 1995 population predicted by the model to the 1995 population in the table?

- **A** It is nearly the same as the population given in the table.
- **B** It is about 1100 less than the population given in the table.
- **C** It is about 4200 greater than the population given in the table.
- **D** It is about 4200 less than the population given in the table.

2 A median-fit line is used to model the data in the table below.

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>7</th>
<th>10</th>
<th>13</th>
<th>17</th>
<th>19</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>8</td>
<td>15</td>
<td>13</td>
<td>16</td>
<td>20</td>
<td>25</td>
<td>25</td>
<td>29</td>
<td>33</td>
</tr>
</tbody>
</table>

If the slope and y-intercept of the line are rounded to the nearest hundredth, which equation best fits the data?

- **A** \( y = 10.00 + 0.96x \)
- **B** \( y = 10.94 + 0.94x \)
- **C** \( y = 9.07 + 1.22x \)
- **D** \( y = 23.86 + 1.04x \)

3 A least-squares regression line is determined for the points plotted on the graph below.

Which of the following numbers is the approximate value of the x-intercept for the least-squares regression line?

- **A** 12.5
- **B** 15
- **C** 17.5
- **D** 20

4 Several samples of apples are weighed, and the data is given in the table.

<table>
<thead>
<tr>
<th>Number of Apples</th>
<th>3</th>
<th>5</th>
<th>4</th>
<th>4</th>
<th>3</th>
<th>5</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight (oz)</td>
<td>32</td>
<td>54</td>
<td>41</td>
<td>44</td>
<td>29</td>
<td>52</td>
<td>19</td>
<td>31</td>
</tr>
</tbody>
</table>

A median-fit line is used to model the data in the table. According to the linear model, how many ounces will a sample of 10 apples weigh? Round to the nearest ounce.

- **A** 100 ounces
- **B** 107 ounces
- **C** 111 ounces
- **D** 115 ounces
**Practice By Standard**
Clarifying Objective MBC.S.2.2

1. A rubber ball is dropped from various heights. For each drop height, the rebound height is recorded. Which statement best describes the relationship between the height of the drop and the rebound height?
   - **A** There is a positive correlation between drop height and rebound height.
   - **B** There is a negative correlation between drop height and rebound height.
   - **C** There is no correlation between drop height and rebound height.
   - **D** The correlation between drop height and rebound height may be positive or negative.

2. Which of the following would most likely have the least correlation strength to the posted speed limit of a road?
   - **A** the time it takes to drive a mile
   - **B** the average speed of the cars
   - **C** the fraction of drivers exceeding the speed limit
   - **D** the revolutions per minute on the cars’ tires

3. Which r-value indicates the weakest strength of correlation?
   - **A** $r = 0.36$
   - **B** $r = 0.19$
   - **C** $r = -0.16$
   - **D** $r = -0.89$

4. Edward calculated the correlation coefficient for the graph below and found a strong linear correlation between $x$ and $y$.

   After adding three points to the graph, he found almost no linear correlation between $x$ and $y$. Which three points did Edward add to the graph?
   - **A** (6, 1), (7, 0), and (8, −1)
   - **B** (6, 2), (7, 4), and (8, 3)
   - **C** (6, 6), (7, 7), and (8, 8)
   - **D** (6, 1), (7, 1), and (8, 1)

5. The table below shows the rating system for mountain biking trails in Western North Carolina.

<table>
<thead>
<tr>
<th>Rating</th>
<th>Difficulty</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>easy</td>
</tr>
<tr>
<td>2</td>
<td>moderate</td>
</tr>
<tr>
<td>3</td>
<td>more difficult</td>
</tr>
<tr>
<td>4</td>
<td>most difficult</td>
</tr>
</tbody>
</table>

Which of the following measurements most likely has the strongest correlation with how a trail is rated?
   - **A** trail length
   - **B** elevation change (climb)
   - **C** $\frac{1}{\text{trail length}}$
   - **D** $\frac{\text{elevation change (climb)}}{\text{trail length}}$
### Practice By Standard

**Clarifying Objective MBC.S.2.3**

1. What is the mean absolute deviation of the following data?
   - $27, 15, 18, 35, 17, 19, 8, 21, 27, 16$
   - **A** 0
   - **B** 5.76
   - **C** 7.62
   - **D** 20.3

2. What is the sum of the squared deviations from the line of best fit for the following data? Round to the nearest whole number.
   - | $x$ | 5.5 | 6.7 | 3.2 | 4.8 | 9 | 6.6 | 4.5 |
   - | $y$ | 55 | 75 | 50 | 35 | 80 | 50 | 20 |
   - **A** 15
   - **B** 81
   - **C** 283
   - **D** 1254

3. Juanita finds the correlation coefficient for the following data.
   - | $x$ | 18 | 32 | 26 | 38 | 16 | 34 | 14 | 36 |
   - | $y$ | 29 | 57 | 45 | 69 | 25 | 61 | 21 | 65 |
   - Using the value of the correlation coefficient, which statement **best** describes the relationship between the SAT math scores and the SAT verbal scores?
   - **A** Since $r = 0.70$, the relationship is moderately linear.
   - **B** Since $r = 0.70$, the relationship is weakly linear.
   - **C** Since $r = 0.88$, the relationship is moderately linear.
   - **D** Since $r = 0.88$, the relationship is strongly linear.
Practice By Standard
Clarifying Objective MBC.S.2.4

1. The Carolina Hurricanes play in the National Hockey League. The table and graph below show the number of goals and assists for 10 team players during the 2008–2009 season.

<table>
<thead>
<tr>
<th>Player</th>
<th>Goals</th>
<th>Assists</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whitney</td>
<td>24</td>
<td>53</td>
</tr>
<tr>
<td>Staal</td>
<td>40</td>
<td>35</td>
</tr>
<tr>
<td>Ruutu</td>
<td>26</td>
<td>28</td>
</tr>
<tr>
<td>Brind' Amour</td>
<td>16</td>
<td>35</td>
</tr>
<tr>
<td>Samsonov</td>
<td>16</td>
<td>32</td>
</tr>
<tr>
<td>Cullen</td>
<td>22</td>
<td>21</td>
</tr>
<tr>
<td>Corvo</td>
<td>14</td>
<td>24</td>
</tr>
<tr>
<td>Babchuk</td>
<td>16</td>
<td>19</td>
</tr>
<tr>
<td>Pitkanen</td>
<td>7</td>
<td>26</td>
</tr>
<tr>
<td>LaRose</td>
<td>19</td>
<td>12</td>
</tr>
</tbody>
</table>

Which line is the least squares regression line? What is its strength as a predictor for the number of assists if a player has 35 goals?

A. Line L; strong predictor
B. Line L; weak predictor
C. Line M; strong predictor
D. Line M; weak predictor

2. The graph below shows the least-squares regression line for data represented by the plotted points.

If the value of the correlation coefficient is 0.02, which statement best describes the fit of the regression line?

A. Given an x-value, the line is a perfect predictor of the related y-value.
B. Given an x-value, the line is a very strong predictor of the related y-value.
C. Given an x-value, the line is a moderate predictor of the related y-value.
D. Given an x-value, the line is a very weak predictor of the related y-value.
Practice By Standard
Clarifying Objective MBC.D.2.1

1. Which linear function maximizes $4x + y$ for the feasible region shown below?
   - the line through point $E$
   - the line through point $F$
   - the line through point $G$
   - the line through point $H$

2. Consider the constraints below.
   
   $x \geq 1$
   $y \leq 4$
   $2x + y \leq 10$
   $x - 2y \geq -5$

   Which ordered pair is a vertex of the feasible region?
   - $(1, 0)$
   - $(1, 3)$
   - $(3, 0)$
   - $(5, 2)$

3. A manufacturer makes two products. The deluxe model requires 5 hours of development time and 3 hours of finishing time, which results in a $45 profit. The economy model requires 3.5 hours of development time and 1.5 hours of finishing time, which results in a $30 profit. Time for development is limited to 50 hours, and time for finishing is limited to 35 hours. How many of each type of product should be developed to maximize profit?
   - 5 deluxe, 6 economy
   - 8 deluxe, 0 economy
   - 10 deluxe, 3 economy
   - 14 deluxe, 0 economy

4. A health food store makes two types of trail mix using a mixture of almonds and cashews. A batch of Mix A contains 4 pounds of almonds and 2 pounds of cashews. A batch of Mix B contains 3 pounds of almonds and 5 pounds of cashews. The store only has 62 pounds of almonds and 80 pounds of cashews. The profit is $4 on each batch of Mix A and $5 on each batch of Mix B. How many batches of each type of mix should be made to maximize profit? What is the maximum profit?
   - 16 batches of Mix A and no batches of Mix B; $64$
   - 10 batches of Mix A and 8 batches of Mix B; $72$
   - 5 batches of Mix A and 14 batches of Mix B; $90$
   - 12 batches of Mix A and 10 batches of Mix B; $98$
Practice By Standard
Clarifying Objective MBC.N.1.1

1 Which expression is equivalent to \( \frac{1}{3} \sqrt[3]{y} \)?
   - A. \( y \)
   - B. \( y^{\frac{1}{3}} \)
   - C. \( \frac{1}{3} y^{\frac{1}{3}} \)
   - D. \( 3y^{\frac{1}{3}} \)

2 A map of Randolph County, North Carolina, is shown below.

RANDOLPH COUNTY
Area = 790 mi²

Randolph County is considered to be a rough square. The distance from the southern border of the county to the northern border is about \( (2.2)790^{\frac{1}{2}} \) kilometers. Which expression represents the same distance?
   - A. \( \sqrt{2.2(790)} \) km
   - B. \( 4.84\sqrt{790} \) km
   - C. \( \sqrt{4.84(790)} \) km
   - D. \( \sqrt{2.2} \sqrt{790} \) km

3 Which process best demonstrates how to simplify \( (-32)^{\frac{1}{3}} \)?
   - A. \( (-32)^{\frac{1}{3}} = \frac{1}{5}(-32) = -6.4 \)
   - B. \( (-32)^{\frac{1}{3}} = \sqrt[3]{-32} = -2 \)
   - C. \( (-32)^{\frac{1}{3}} = \sqrt[3]{32} = 2 \)
   - D. \( (-32)^{\frac{1}{3}} = \sqrt[3]{-32} \approx -5793 \)

4 If \( f(x) = \left( -\frac{1}{x} \right)^{\frac{1}{3}} \), what is \( f(3) \)?
   - A. \( -\frac{\sqrt[3]{3}}{3} \)
   - B. \( -\frac{1}{\sqrt[3]{3}} \)
   - C. \( -\frac{1}{243} \)
   - D. \( -27 \)

5 Which of the following is equivalent to the expression \( \frac{8^{\frac{1}{2}} \cdot 8^{\frac{1}{2}}}{2^{\frac{1}{2}}} \)?
   - A. 8
   - B. \( 8\sqrt{2} \)
   - C. \( \frac{2}{\sqrt{2}} \)
   - D. \( 4\sqrt{2} \)

Go on
Practice By Standard
Clarifying Objective MBC.N.1.2

1. Which expression is equivalent to 
   \((27x^3)^{\frac{1}{2}}\) for all values of \(x\)?
   - A: \(3x\sqrt[3]{3x}\)
   - B: \(3|x|\sqrt[3]{3x}\)
   - C: \(9x\sqrt[3]{3x}\)
   - D: \(9|x|\sqrt[3]{3x}\)

2. Simplify \(\frac{r^\frac{1}{2} - z^\frac{1}{2}}{r^\frac{1}{3} - z^\frac{1}{3}}\).
   - A: \(\frac{1}{r^\frac{1}{3} - z^\frac{1}{3}}\)
   - B: \(\frac{1}{r^\frac{1}{3} + z^\frac{1}{3}}\)
   - C: \(r^\frac{1}{3} - z^\frac{1}{3}\)
   - D: \(r^\frac{1}{3} + z^\frac{1}{3}\)

3. Simplify \(\left[243(x^{15}y^{19})^2\right]^\frac{1}{3}\).
   - A: \(9x^6y^2\)
   - B: \(9x^2y^2\)
   - C: \(3x^{30}y^{20}\)
   - D: \(3x^6y^4\)

4. Solve \(4^{\frac{6x}{3}} + 4 = 16^{\frac{1}{3}}\).
   - A: \(x = \frac{1}{2}\)
   - B: \(x = \frac{1}{4}\)
   - C: \(x = 0\)
   - D: \(x = \frac{1}{5}\)

5. What is the simplest form of
   \(\frac{(a^\frac{3}{2}b^\frac{1}{2}c^\frac{1}{2})^4}{(a^3b^2c^3)^{\frac{3}{2}}}\)?
   - A: \(\frac{c^4}{4a^3}\)
   - B: \(\frac{c^3}{4a^3b}\)
   - C: \(\frac{bc^3}{4a^3}\)
   - D: \(\frac{a^3b^2c^5}{2}\)

6. Simplify \(\frac{8a^{-2}b^{\frac{1}{3}}}{12x^{-3}y^{-\frac{1}{3}}} \div \frac{4a^{-1}b^{-\frac{2}{3}}}{3x^{-4}y^{-1}}\).
   - A: \(\frac{9x^2y^4}{8a^2b^2}\)
   - B: \(\frac{b^{\frac{1}{3}}y^2}{2ax^2}\)
   - C: \(\frac{8b^{\frac{1}{3}}y^2}{9ax^2}\)
   - D: \(\frac{by^2}{2ax^2}\)
1. In Asheville, North Carolina, the average high temperatures for December, January, and February are 50°, 46°, and 50°. For the same months in Cape Hatteras, North Carolina, the average high temperatures are 57°, 53°, and 54°. Which is a possible matrix for the high temperatures of these two locations for the given months?

   A) \[
   \begin{bmatrix}
   50 & 46 & 50 \\
   44 & 40 & 41 \\
   50 & 53 \\
   57 & 50 \\
   46 & 54 \\
   
   \end{bmatrix}
   \]

   B) \[
   \begin{bmatrix}
   50 & 53 \\
   46 & 54 \\
   
   \end{bmatrix}
   \]

   C) \[
   \begin{bmatrix}
   50 & 46 & 50 \\
   57 & 53 & 54 \\
   50 & 57 & 46 \\
   53 & 50 & 54 \\
   
   \end{bmatrix}
   \]

   D) \[
   \begin{bmatrix}
   50 & 46 & 50 \\
   57 & 53 & 54 \\
   50 & 57 & 46 \\
   
   \end{bmatrix}
   \]

2. Terrell is shopping for shirts at different clothing stores. He creates the matrix of prices below.

\[
\begin{bmatrix}
37.95 & 31.50 & 29 & 39.99 \\
19.98 & 27.49 & 21.50 & 27.97
\end{bmatrix}
\]

Which type of data is most likely represented by the matrix?

   A) 2 clothing stores and 4 shirt brands
   B) 8 clothing stores and 8 shirt brands
   C) 8 clothing stores and 1 shirt brand
   D) 1 clothing store and 8 shirt brands

3. Consider the line graphed below.

Which of the following matrices could be used to display ordered pairs from the line?

   A) \[
   \begin{bmatrix}
   4 & 2 & 4 \\
   0 & 1 & -2
   \end{bmatrix}
   \]

   B) \[
   \begin{bmatrix}
   -2 & 7 & -1 & 5.5 \\
   0 & 4 & 1 & 2.5
   \end{bmatrix}
   \]

   C) \[
   \begin{bmatrix}
   0 & 3 & 4 \\
   4 & -0.5 & 2
   \end{bmatrix}
   \]

   D) \[
   \begin{bmatrix}
   -1 & 1 & 3 \\
   5.5 & 2.5 & -0.5
   \end{bmatrix}
   \]

4. Which of the following could be true of a matrix that contains 12 elements?

   A) The matrix has 6 rows and 6 columns.
   B) The matrix has 5 rows.
   C) The matrix has 7 columns.
   D) The matrix has 3 rows and 4 columns.
Practice By Standard
Clarifying Objective MBC.N.2.2

1. The North Carolina state sales tax is 4.5%. If a matrix contains item prices, what matrix operation would result in the prices with state sales tax included?
   A. adding a matrix where each element is 0.045
   B. multiplying the matrix by a scalar of 0.045
   C. multiplying the matrix by a scalar of 1.045
   D. multiplying the matrix by a scalar of 4.5

2. Consider the matrices
   \[ K = \begin{bmatrix} 2 & 5.1 \\ 1 & 3.2 \end{bmatrix} \]
   and
   \[ L = \begin{bmatrix} -1 & 0 \\ 2.5 & -1.8 \end{bmatrix} \]
   What is \( K + L \)?
   A. \[ \begin{bmatrix} 1 & 5.1 \\ 3.5 & 1.4 \end{bmatrix} \]
   B. \[ \begin{bmatrix} 3 & 5.1 \\ -1.5 & 5 \end{bmatrix} \]
   C. \[ \begin{bmatrix} 3 & 5.1 \\ 3.5 & -5 \end{bmatrix} \]
   D. \[ \begin{bmatrix} 1 & 1 \\ 7.6 & 1.4 \end{bmatrix} \]

3. Matrix \( G \) has 36 elements. Which of the following ensures that \( 2G - H \) exists for some matrix \( H \)?
   A. Matrix \( H \) has 36 elements.
   B. Matrix \( H \) has 72 elements.
   C. Matrix \( H \) has the same number of rows and columns as matrix \( G \).
   D. Matrix \( H \) has the same number of rows as the number of columns in matrix \( G \).

4. Matrices \( E \) and \( F \) are shown below.
   \[ E = \begin{bmatrix} 1 & -3 & 6 \\ -4 & 0 & -8 \\ 3 & 9 & -1 \end{bmatrix} \]
   \[ F = \begin{bmatrix} -2 & -1 & 3 \\ 5 & 7 & -4 \\ -2 & -6 & 1 \end{bmatrix} \]
   What is \( -3F - 2E \)?
   A. \[ \begin{bmatrix} 1 & 11 & -24 \\ 2 & -14 & 32 \\ -5 & -15 & 1 \end{bmatrix} \]
   B. \[ \begin{bmatrix} 4 & 9 & -21 \\ -7 & -21 & 28 \\ 0 & 0 & -1 \end{bmatrix} \]
   C. \[ \begin{bmatrix} 8 & -3 & 3 \\ -23 & -21 & -4 \\ 12 & 36 & -5 \end{bmatrix} \]
   D. \[ \begin{bmatrix} -8 & 3 & -3 \\ 23 & 21 & 4 \\ -12 & -36 & 5 \end{bmatrix} \]

5. For two matrices \( I \) and \( J \), the sum of the matrices \( I + J \) is \[ \begin{bmatrix} 7 & 3 \\ 6 & -4 \end{bmatrix} \]. The difference of \( I - J \) is \[ \begin{bmatrix} 1 & 5 \\ -4 & 10 \end{bmatrix} \]. What is Matrix \( J \)?
   A. \[ \begin{bmatrix} 8 & -2 \\ 10 & 6 \end{bmatrix} \]
   B. \[ \begin{bmatrix} 6 & -2 \\ 10 & -6 \end{bmatrix} \]
   C. \[ \begin{bmatrix} 3 & 4 \\ 1 & -7 \end{bmatrix} \]
   D. \[ \begin{bmatrix} 3 & -1 \\ 5 & -7 \end{bmatrix} \]
Practice By Standard
Clarifying Objective MBC.N.2.3

1 Suppose $M$ is a $5 \times 3$ matrix, $N$ is a $4 \times 3$ matrix, and $P$ is a $4 \times 4$ matrix. Which matrix product exists?
- [A] $MN$
- [B] $MP$
- [C] $NP$
- [D] $PN$

2 The Duke Blue Devils played the North Carolina Tar Heels in an NCAA basketball game in March of 2008. In the matrices below, LF represents successful long field goals, SF represents successful short field goals, FT represents successful free throws, and PT is the number of points for each type of shot.

<table>
<thead>
<tr>
<th></th>
<th>LF</th>
<th>SF</th>
<th>FT</th>
<th>PT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duke</td>
<td>10</td>
<td>15</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>N.C.</td>
<td>5</td>
<td>26</td>
<td>9</td>
<td>1</td>
</tr>
</tbody>
</table>

What is the product matrix? What is its real-world meaning?
- [A] $[35 \ 108 \ 43]$; the number of points from each type of successful shot
- [B] $[68 \ 76]$; the number of points scored by all successful shots by Duke and North Carolina, respectively
- [C] $[67 \ 60]$; the number of points from field goals and the number of points from free throws
- [D] $[33 \ 40]$; the combined number of successful goals and free throws for Duke and North Carolina, respectively

3 Given the matrices:

$$Q = \begin{bmatrix} 3 & 2 \\ -1 & 0 \\ 4 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} -1 & -2 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

What is $QR$?
- [A] $\begin{bmatrix} 1 & -4 & -4 \\ -2 & 7 & -2 \end{bmatrix}$
- [B] $\begin{bmatrix} 1 & -10 & 0 \\ -5 & -10 & -2 \end{bmatrix}$
- [C] $\begin{bmatrix} -1 & 2 \\ -3 & 2 \end{bmatrix}$
- [D] $\begin{bmatrix} -7 & 0 \\ -6 & 9 \end{bmatrix}$

4 Felicia sells lunches at three different food carts around town. Each cart offers lunch in small, medium, and large portions. The matrices below show her prices for each lunch and the number of lunches she sold yesterday.

<table>
<thead>
<tr>
<th></th>
<th>Sm</th>
<th>Med</th>
<th>Lg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prices ($)</td>
<td>[2.75 3.50 4.00]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number Sold:</th>
<th>cart</th>
<th>cart</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>12</td>
<td>15</td>
</tr>
<tr>
<td>B</td>
<td>10</td>
<td>24</td>
</tr>
<tr>
<td>C</td>
<td>8</td>
<td>20</td>
</tr>
</tbody>
</table>

Felicia multiplies the matrices to find matrix $S$, her total sales in dollars at each cart. What is $S_{13}$?
- [A] $100$
- [B] $140$
- [C] $155$
- [D] $205.25$
An open garage door starts at a height of 7 feet and closes at a rate of 7 inches per second. A sensor will reverse the direction of the door if an object is encountered while the door is moving.

Which equation results in the graph shown for the domain \( \{x | 0 \leq x \leq 12\} \)?

- **A** \( y = |x| + 1 \)
- **B** \( y = |7 - x| + 1 \)
- **C** \( y = |x - 6| + 1 \)
- **D** \( y = \begin{cases} 7 - x & \text{if } x < 7 \\ x & \text{if } x \geq 7 \end{cases} \)

Which equation is graphed on the coordinate grid shown below?

- **A** \( y = -x^2 - 2x + 1 \)
- **B** \( y = -x^2 + 4x - 3 \)
- **C** \( y = -x^2 - 4x + 3 \)
- **D** \( y = x^2 + 2x + 1 \)

The function \( y = [x] \) represents the greatest integer function, where \([x]\) returns the nearest integer that is less than or equal to \( x \). Which is the graph of \( y = [x] \)?
Juanita wants to deposit her savings in an account that compounds interest. If she leaves the money in the account for 2 years, the final value of her deposit is a function of the interest rate. The relationship is shown in the table.

<table>
<thead>
<tr>
<th>r</th>
<th>F(r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>$260.10</td>
</tr>
<tr>
<td>3%</td>
<td>$265.23</td>
</tr>
<tr>
<td>4%</td>
<td>$270.40</td>
</tr>
<tr>
<td>5%</td>
<td>$275.63</td>
</tr>
</tbody>
</table>

Which function does the table represent?

A. \( F(r) = 250(1 + r)^2 \)
B. \( F(r) = 250(1 - r)^2 \)
C. \( F(r) = 250r^2 \)
D. \( F(r) = 250 + (1 + r)^2 \)

The number of handshakes \( h \) in a room with \( n \) people is given in the table.

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>h</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

Which function best describes the relationship given in the table?

A. \( h = n - 1 \)
B. \( h = n(n - 1) \)
C. \( h = \frac{1}{2}(n^2 - n) \)
D. \( h = \frac{n^2}{2} - n \)

The cubes of whole numbers are given in the table.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>1</td>
<td>8</td>
<td>27</td>
<td>64</td>
<td>125</td>
</tr>
</tbody>
</table>

Which function describes the difference \( d(x) \) between \( y \) and the next value of \( y \) in the table?

A. \( d(x) = x^3 \)
B. \( d(x) = 6x - 5 \)
C. \( d(x) = x^3 - x \)
D. \( d(x) = 3x^2 + 3x + 1 \)

What function is graphed below?

\[
A \quad y = \begin{cases} 
1 + x^2 & \text{if } x < 1 \\
(x - 1)^2 & \text{if } x \geq 1 
\end{cases}
B \quad y = \begin{cases} 
1 - x^2 & \text{if } x < 1 \\
x^2 - 1 & \text{if } x \geq 1 
\end{cases}
C \quad y = \begin{cases} 
1 - x^2 & \text{if } x < 1 \\
2x - 1 & \text{if } x \geq 1 
\end{cases}
D \quad y = \begin{cases} 
1 + x^2 & \text{if } x < 1 \\
x^2 & \text{if } x \geq 1 
\end{cases}
\]
Practice By Standard
Clarifying Objective MBC.A.2.1

1. What matrix equation can be used to solve the system of equations below?

\[
\begin{align*}
4x &= 5y \\
-2x - 3y &= 8
\end{align*}
\]

A. \[
\begin{bmatrix}
4 & 5 \\
-2 & 3
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
5 \\
8
\end{bmatrix}
\]

B. \[
\begin{bmatrix}
4 & 5 \\
2 & -3
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
0 \\
8
\end{bmatrix}
\]

C. \[
\begin{bmatrix}
4 & -5 \\
-2 & 3
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
0 \\
8
\end{bmatrix}
\]

D. \[
\begin{bmatrix}
4 & 0 \\
-2 & -3
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
5 \\
8
\end{bmatrix}
\]

2. Which process can be used to solve a matrix equation?

A. Multiply the coefficient matrix by the variable matrix.

B. Multiply the inverse of the coefficient matrix by the variable matrix.

C. Multiply the constant matrix by the variable matrix.

D. Multiply the inverse of the coefficient matrix by the constant matrix.

3. Solve \[
\begin{bmatrix}
-2 & 5 \\
3 & 7
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
-21 \\
-12
\end{bmatrix}
\]

A. \( x = \frac{21}{29}, y = \frac{12}{29} \)

B. \( x = 3, y = -3 \)

C. \( x = -3, y = 3 \)

D. \( x = \frac{-111}{29}, y = \frac{129}{29} \)

4. Which matrix equation is the solution to the system of equations?

\[
\begin{align*}
x - 3y &= 7 \\
2x - y &= 5
\end{align*}
\]

A. \[
\begin{bmatrix}
1 & 3 \\
2 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
7 \\
5
\end{bmatrix}
\]

B. \[
\begin{bmatrix}
-1 & -2 \\
5 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
7 \\
5
\end{bmatrix}
\]

C. \[
\begin{bmatrix}
-1 & -2 \\
-3 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
7 \\
5
\end{bmatrix}
\]

D. \[
\begin{bmatrix}
-1 & 3 \\
5 & -2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
7 \\
5
\end{bmatrix}
\]

5. The number of adult, discount, and infant admissions tickets sold over three days, to the Asheville Art Museum, and the total amount of ticket sales for those days are shown in the matrix equation below.

\[
\begin{bmatrix}
342 & 316 & 24 \\
421 & 380 & 16 \\
377 & 408 & 18
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
3632 \\
4426 \\
4302
\end{bmatrix}
\]

What was the price of admission for each type of ticket?

A. adult: $7, discount: $4, infant: $1

B. adult: $8, discount: $6, infant: free

C. adult: $6, discount: $5, infant: free

D. adult: $6, discount: $3.50, infant: $1
**Practice By Standard**

**Clarifying Objective MBC.A.2.2**

1. Solve
   
   \[ \begin{align*}
   2a - 3b &= -2 \\
   -2a + b &= -6.
   \end{align*} \]

   **A** \( a = 5; \ b = 4 \)
   **B** \( a = \frac{1}{5}; \ b = \frac{1}{4} \)
   **C** infinitely many solutions
   **D** no solution

2. Which system of inequalities has no solution?

   **A** \( \begin{align*}
   2x + y &> 0 \\
   3x + y &\leq 0
   \end{align*} \)
   **B** \( \begin{align*}
   2y - x &> 2 \\
   2y - x &< 6
   \end{align*} \)
   **C** \( \begin{align*}
   3x + y &> -1 \\
   x - 3y &> 13
   \end{align*} \)
   **D** \( \begin{align*}
   4y - 2x &> 12 \\
   2y - x &< 6
   \end{align*} \)

3. What is the solution to the system of equations shown below?

   \[ \begin{align*}
   9x + 3y + 3z &= -12 \\
   27x + 9y + 9z &= -36 \\
   -3x - y - z &= 4
   \end{align*} \]

   **A** \( (0, 0, \frac{1}{4}) \)
   **B** \( (0, 4, 0) \)
   **C** infinitely many solutions
   **D** no solution

4. The state vegetable of North Carolina is the sweet potato. The table below, on the left, shows the amount of each energy source of three different serving sizes of sweet potato. The table below, on the right, shows the total energy in each serving size.

<table>
<thead>
<tr>
<th>Carb. (g)</th>
<th>Fat (g)</th>
<th>Protein (g)</th>
<th>Energy (Cal.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>23.6</td>
<td>0.2</td>
<td>2.3</td>
<td>114</td>
</tr>
<tr>
<td>47.2</td>
<td>0.4</td>
<td>4.6</td>
<td>228</td>
</tr>
<tr>
<td>70.8</td>
<td>0.6</td>
<td>6.9</td>
<td>342</td>
</tr>
</tbody>
</table>

   Can a system of equations from the data be used to find the Calories contributed by each energy source?

   **A** Yes, you can set up 3 equations for the 3 unknown values and solve.
   **B** No, the system is inconsistent.
   **C** No, the system is dependent.
   **D** No, the system would need to be a system of inequalities.

5. What system of inequalities best represents the graph shown below?

   **A** \( 2x + 3y < 0 \) and \( x - y \geq 5 \)
   **B** \( 2x + 3y \geq 0 \) and \( x - y < 5 \)
   **C** \( 2x + 3y < 0 \) and \( x - y \leq 5 \)
   **D** \( 2x + 3y > 0 \) and \( x - y \leq 5 \)
Which best describes the transformation of the parent function \( f(x) = |x| \) to \( g(x) = - |x - 1| \)?

A. The graph is shifted 1 unit right and reflected across the x-axis.
B. The graph is shifted 1 unit left and reflected across the x-axis.
C. The graph is shifted 1 unit right and reflected across the y-axis.
D. The graph is shifted 1 unit left and reflected across the y-axis.

Which translation of the graph of \( f(x) = 2|x| \) shares at least two points with function \( f(x) \)?

A. \( g(x) = |x| \)
B. \( g(x) = |x - 1| - 2 \)
C. \( g(x) = 2|x - 1| + 2 \)
D. \( g(x) = 2|x| + 1 \)

Which transformations of the function \( f(x) = \left(\frac{1}{2}\right)^x \) are equivalent?

A. \( g(x) = 2\left(\frac{1}{2}\right)^x \) and \( h(x) = \left(\frac{1}{2}\right)^{x-1} \)
B. \( g(x) = 2\left(\frac{1}{2}\right)^x \) and \( h(x) = \left(\frac{1}{2}\right)^{x+1} \)
C. \( g(x) = 2^{x+1} \) and \( h(x) = \left(\frac{1}{2}\right)^{x-1} \)
D. \( g(x) = 2\left(\frac{1}{2}\right)^{x+1} \) and \( h(x) = \left(\frac{1}{2}\right)^{x-1} \)

In the general transformation of the parent quadratic function \( f(x) = x^2 \) to \( g(x) = a(x - h)^2 + k \), which value(s) will affect the location of the vertex?

A. \( h \) and \( k \)
B. \( a \) and \( h \)
C. \( a \) only
D. \( h \) only

Which description most accurately describes the translation of the graph \( f(x) = 2(x - 1)^2 - 5 \) to the graph of \( g(x) = 2(x - 2)^2 - 3 \)?

A. 2 units right
B. 2 units up and 1 unit right
C. 2 units up and 1 unit left
D. 1 unit up and 2 units right

Which of the following transformations shifts the function \( f(x) \), 3 units left and 2 units down?

A. \( f(x) = (x - 1)^2 \) and \( g(x) = 2 - (x + 2)^2 \)
B. \( f(x) = (x + 1)^2 \) and \( g(x) = (x - 2)^2 - 2 \)
C. \( f(x) = (x - 1)^2 \) and \( g(x) = (x + 2)^2 - 2 \)
D. \( f(x) = x^2 \) and \( g(x) = (x + 3)^2 - 2 \)
Practice By Standard
Clarifying Objective MBC.A.3.2

1 Camila reflected and translated \( f(x) = (x + 4)^2 + 1 \) to make a new function. The graph below shows \( f \) and its transformation.

Which equation represents the transformed function?

A \( g(x) = (\text{-}x)^2 + 1 \)

B \( g(x) = \text{-}x^2 - 1 \)

C \( g(x) = (x - 4)^2 - 1 \)

D \( g(x) = (\text{-}x + 4)^2 + 5 \)

2 Which transformation of \( f(x) = x^2 \) results in the widest curve?

A \( g(x) = 2(x + 10)^2 + 3 \)

B \( g(x) = \text{-}\frac{5}{2}(x - 2)^2 - 4 \)

C \( g(x) = \frac{7}{3}(x - 5)^2 + 20 \)

D \( g(x) = \frac{(x - 1)^2}{5} \)

3 If the graph of the equation \( f(x) = \text{-}(x + 1)^2 + 2 \) is shifted 1 unit left, which would be the graph of the transformed parabola?

STOP
Practice Test

1. For their school trip this year, Mrs. Callahan’s students decided to visit one state park and one historical battleground. Their state park choices are Fort Macon, Goose Creek, and Fort Fisher. The battleground choices are Almanac Battleground and Bettonville Battlefield. Which is a diagram of the sample space for the class field trip?

2. Consider the matrices below.

\[ S = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & 4 \end{bmatrix} \quad T = \begin{bmatrix} 5 & 1 \\ -2 & 4 \\ 6 & 0 \end{bmatrix} \]

What is the product of \( S \) and \( T \)?

A. \[ \begin{bmatrix} -10 & 3 & 9 \\ 4 & 12 & 14 \end{bmatrix} \]
B. \[ \begin{bmatrix} -4 & -2 \\ 18 & 12 \end{bmatrix} \]
C. \[ \begin{bmatrix} -10 & 0 \\ 0 & 12 \end{bmatrix} \]
D. \[ \begin{bmatrix} -12 & 0 & 5 \\ 0 & 12 & 4 \end{bmatrix} \]

3. What are the \( x \)-intercepts of the graph of the equation \( f(x) = 2x^2 - 4x + 1 \)?

A. \( \frac{2 + \sqrt{2}}{2} \) and \( \frac{2 - \sqrt{2}}{2} \)
B. \( \frac{2 + \sqrt{6}}{2} \) and \( \frac{2 - \sqrt{6}}{2} \)
C. \( 1 + 2\sqrt{2} \) and \( 1 - 2\sqrt{2} \)
D. \( -1 + 2\sqrt{2} \) and \( -1 - 2\sqrt{2} \)
Practice Test (continued)

4 Reggie uses a compass and a straightedge to draw a line containing point $P$ which is parallel to $\overline{MN}$ as shown below.

![Diagram of a line containing point P parallel to MN]

Which statement best supports Reggie’s construction?

A. If consecutive interior angles are supplementary, then lines are parallel.
B. If alternate interior angles are congruent, then lines are parallel.
C. Parallel lines have the same slope.
D. Through any point not on a line, there is exactly one line parallel to the given line.

5 If $f(x) = 4x^2 - x$ and $g(x) = 3x^2$, which is an equivalent form of $g(x) - f(x)$?

A. $g(x) - f(x) = -x^2 - x$
B. $g(x) - f(x) = -x^2 + x$
C. $g(x) - f(x) = x^2 - x$
D. $g(x) - f(x) = x^2 + x$

6 The center of a circle is located at (4, 2). If A(3, 5) is a point on the circle, which equation represents the circle?

A. $(x - 3)^2 + (y - 5)^2 = 100$
B. $(x - 4)^2 + (y - 2)^2 = 100$
C. $(x - 3)^2 + (y - 5)^2 = 10$
D. $(x - 4)^2 + (y - 2)^2 = 10$

7 Don is making a pet food mixture by combining ingredient $H$ with ingredient $J$. Don wants to make at least 5 pounds of pet food that contains at least 60 grams of protein. Ingredient $H$ has 8 grams of protein per pound, and Ingredient $J$ has 16 grams of protein per pound. If ingredient $H$ costs $4 per pound and ingredient $J$ costs $3 per pound, what is the minimum cost of the pet food?

A. $14.00$
B. $17.50$
C. $20.00$
D. $22.50$

8 Which function has a maximum at $x = 2$?

A. $f(x) = -4x^2 + 8x - 3$
B. $f(x) = -2x^2 + 8x + 1$
C. $f(x) = x^2 - 4x - 7$
D. $f(x) = 3x^2 + 12x + 10$
9. Lines $q$ and $r$ are parallel in the figure below.

If $\angle 3 \equiv \angle 9$, which relationship is true?

A. $\angle 4 \equiv \angle 16$
B. $\angle 1 \equiv \angle 7$
C. $\angle 5 \equiv \angle 11$
D. $\angle 4 \equiv \angle 15$

10. The graph of $f(x) = |x|$ is translated 3 units up and 2 units left. Which equation is represented by the translated graph?

A. $g(x) = |x + 2| + 3$
B. $g(x) = |x - 2| + 3$
C. $g(x) = |x + 2| - 3$
D. $g(x) = |x - 2| - 3$

12. Simplify $\log_4 x^3 y^5$.

A. $8 \log_4 xy$
B. $15 \log_4 xy$
C. $3 \log_4 x + 5 \log_4 y$
D. $8 + \log_4 x + \log_4 y$

13. Which matrix is equivalent to $\begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix} - 2 \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix}$?

A. $\begin{bmatrix} 4 & 0 \\ -3 & -1 \end{bmatrix}$
B. $\begin{bmatrix} 4 & 0 \\ 9 & -1 \end{bmatrix}$
C. $\begin{bmatrix} -4 & 0 \\ -3 & -1 \end{bmatrix}$
D. $\begin{bmatrix} -4 & 0 \\ 9 & -1 \end{bmatrix}$

14. Brett finds the median-fit line for a set of data. The three medians are $(8, 24), (17, 19)$, and $(26, 12)$. What is the equation of the median-fit line?

A. $y = 29.33 - 0.67x$
B. $y = 29.5 - 0.67x$
C. $y = 29.67 - 0.67x$
D. $y = 30 - 0.67x$
Practice Test (continued)

15 Which is a pair of similar figures?

A

B

C

D

16 What is the vertical shift when transforming \( y = \cos x \) into \( y = \cos \left( x - \frac{\pi}{4} \right) + 3 \) ?

A \( \frac{\pi}{4} \) unit up

B \( \frac{\pi}{4} \) unit down

C 3 units up

D 3 units down

17 The height in feet of a kicked football is represented by the function \( f(t) = -16t^2 + 64t + 2 \), where \( t \) is the time in seconds. What is the maximum height reached by the football?

A 32 feet

B 50 feet

C 64 feet

D 66 feet

18 North Carolina’s Research Triangle is an area known for its high technology businesses, many of which have benefited from exponential improvements in computer hardware. Moore’s law states that the number of transistors that can inexpensively be placed on an integrated circuit doubles every two years. Since the number of transistors on a circuit was 2300 in 1971, the equation \( n = 2300 \times 2^\frac{y}{2} \) can be used to predict the number of transistors \( n \) on a circuit \( y \) years after 1971. Which logarithmic equation is equivalent to \( n = 2300 \times 2^\frac{y}{2} \) ?

A \( y = \log_n 2300 \)

B \( y = \frac{1}{2} \log_n 2300 \)

C \( y = \log_2 \left( \frac{n}{2300} \right) \)

D \( y = 2 \log_2 \left( \frac{n}{2300} \right) \)
19. Which is a valid conclusion drawn from the true conditional statement in the box below?

If Katia is in the capital of North Carolina, then she is in Raleigh.

A. If Katia is in North Carolina, then she is in the state capital.
B. If Katia is in North Carolina, then she is not in Raleigh.
C. If Katia is not in Raleigh, then she is not in North Carolina.
D. If Katia is not in the state capital of North Carolina, then she is not in Raleigh.

20. Which matrix equation represents the system of equations below?

\[
\begin{align*}
-2y + z &= 4 \\
-x + 3y &= 0 \\
2x + 5z &= 8
\end{align*}
\]

A. \[
\begin{bmatrix}
-2 & 1 & 0 \\
-1 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
4 \\
0 \\
8
\end{bmatrix}
\]
B. \[
\begin{bmatrix}
0 & -2 & 1 \\
-1 & 3 & 0 \\
2 & 5 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
4 \\
0 \\
8
\end{bmatrix}
\]
C. \[
\begin{bmatrix}
0 & -2 & 1 \\
-1 & 3 & 0 \\
2 & 5 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
4 \\
0 \\
8
\end{bmatrix}
\]
D. \[
\begin{bmatrix}
0 & -2 & 1 \\
-1 & 3 & 0 \\
2 & 0 & 5
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
4 \\
0 \\
8
\end{bmatrix}
\]

21. What is the simplest form of \[
\frac{1}{x-3} \div \frac{1}{x-3}
\]

A. 1
B. \(\frac{x-4}{x-3^2}\)
C. \(\frac{1}{x-4}\)
D. \(\frac{1}{x-2}\)

22. Adam is one of 7 students whose names are in a bag. Two names will be selected from the bag at random to win a prize. The first student chosen is Kelly. Which statement is true about Adam’s chances of winning?

A. Adam has the same chance of winning a prize whether Kelly’s name is returned to the bag or not.
B. Adam has a better chance of winning if Kelly’s name is returned to the bag.
C. Adam has a better chance of winning if Kelly’s name is kept out of the bag.
D. Adam’s chances of winning are greater than those of the other remaining students.
23. Kai wants to simplify \( \frac{x^2 - 1}{x^2 - 4} \cdot \frac{3x + 6}{4x} \) before multiplying. Which expression is equivalent to \( \frac{x^2 - 1}{x^2 - 4} \cdot \frac{3x + 6}{4x} \)?

- **A**: \( \frac{1}{2} \cdot \frac{3}{4x} \)
- **B**: \( \frac{(x - 1)}{(x - 2)} \cdot \frac{3}{4x} \)
- **C**: \( \frac{(x + 1)(x - 1)}{(x - 2)} \cdot \frac{3}{4x} \)
- **D**: \( \frac{(x + 1)(x - 1)}{(x - 2)} \cdot \frac{3(x + 2)}{4x} \)

24. A circle is graphed on the coordinate plane. The diameter of the circle has endpoints at \((-4, 5)\) and \((3, -7)\). What is the equation of the circle?

- **A**: \( (x - \frac{1}{2})^2 + (y - 1)^2 = 48.25 \)
- **B**: \( (x + \frac{1}{2})^2 + (y + 1)^2 = 48.25 \)
- **C**: \( (x - 1)^2 + (y - 2)^2 = 56 \)
- **D**: \( (x + 1)^2 + (y + 2)^2 = 56 \)

25. Given:

I. The sum of the interior angles of a polygon with 3 sides equals 180°.
II. The sum of the interior angles of a polygon with 4 sides equals 360°.
III. The sum of the interior angles of a polygon with 5 sides equals 540°.
IV. The sum of the interior angles of a polygon with 6 sides equals 720°.

What conclusion can be inferred based on the given information?

- **A**: When a polygon has \( n \) sides, the sum of the interior angles will be \( 360n° \).
- **B**: When a polygon has \( n \) sides, the sum of the interior angles will be \( 180n° \).
- **C**: When a polygon has \( n \) sides, the sum of the interior angles will be \( 360(n - 2)° \).
- **D**: When a polygon has \( n \) sides, the sum of the interior angles will be \( 180(n - 2)° \).

26. Which transformation results in a congruent figure?

- **A**: a dilation by a factor of 3 and a 90° clockwise rotation about the origin
- **B**: a reflection across the line \( y = -1 \) and a translation up 3 units
- **C**: a translation down 2 units and a dilation by a factor of 2
- **D**: a 270° rotation about the origin and a dilation by a factor of 3
27 Kalisha is proving that the angle bisectors of \( \triangle LMN \) meet at a single point \( Z \). She begins by constructing angle bisectors \( \overline{NX} \) and \( \overline{MY} \). Next, she constructs perpendicular segments \( \overline{ZQ} \), \( \overline{ZR} \), and \( \overline{ZS} \). Her work is shown below.

Which reasoning could Kalisha use to show that \( \overline{LZ} \) is also an angle bisector?

A \( \triangle QZM \cong \triangle RZM \) and \( \triangle RZN \cong \triangle MZN \) by \( \text{SAS} \), so \( \overline{ZQ} \cong \overline{ZR} \cong \overline{ZS} \). Angles opposite congruent sides are congruent, so \( \angle QLZ \cong \angle SLZ \). Therefore, \( \overline{LZ} \) is an angle bisector.

B \( \triangle XZM \cong \triangle MZN \) and \( \triangle MZN \cong \triangle YZN \) by \( \text{AAS} \), so \( \overline{ZX} \cong \overline{ZY} \). Angles opposite congruent sides are congruent, so \( \angle XLZ \cong \angle YLZ \). Therefore, \( \overline{LZ} \) is an angle bisector.

C \( \triangle XZM \cong \triangle MZN \) and \( \triangle MZN \cong \triangle YZN \) by \( \text{AAS} \), so \( \overline{ZX} \cong \overline{ZY} \). \( \triangle XLZ \cong \triangle YLZ \) by \( \text{HL congruence} \), so \( \angle XLZ \cong \angle YLZ \). Therefore, \( \overline{LZ} \) is an angle bisector.

D \( \triangle QZM \cong \triangle RZM \) and \( \triangle RZN \cong \triangle MZN \) by \( \text{SAS} \), so \( \overline{ZQ} \cong \overline{ZR} \cong \overline{ZS} \). \( \triangle QLZ \cong \triangle SLZ \) by \( \text{HL congruence} \), so \( \angle QLZ \cong \angle SLZ \). Therefore, \( \overline{LZ} \) is an angle bisector.

28 What function is represented by the graph below?

\[
\begin{align*}
y &= \frac{x}{2} \quad \text{if } x < 0 \\
y &= x - 2 \quad \text{if } x \geq 0
\end{align*}
\]

A \( y = \begin{cases} \frac{x}{2} & \text{if } x < 0 \\ x - 2 & \text{if } x \geq 0 \end{cases} \)

B \( y = \begin{cases} \frac{1}{2x} & \text{if } x < 0 \\ 2x - 2 & \text{if } x \geq 0 \end{cases} \)

C \( y = \begin{cases} \frac{x}{2} - 2 & \text{if } x < 0 \\ 2(x - 2) & \text{if } x \geq 0 \end{cases} \)

D \( y = \begin{cases} \frac{1}{2x} - 2 & \text{if } x < 0 \\ 2(x - 1) & \text{if } x \geq 0 \end{cases} \)

29 Which is the best example of an undefined term?

A theorem
B postulate
C point
D angle
30 Given: \( m\overline{QT} = 132^\circ \) and \( \overline{QT} \cong \overline{ST} \)

What is \( m\angle QTS \)?

A 24°  
B 48°  
C 66°  
D 114°

31 Which function could be represented by the graph below?

A \( y = \sqrt{x} \)  
B \( y = \frac{1}{x} \)  
C \( y = \sin x \)  
D \( y = x^4 \)

32 The figure below is from a proof of the Pythagorean Theorem.

Which statement could be used in the proof of the Pythagorean Theorem?

A The area of the inner square is equal to half the area of the larger square.  
B The diagonals of the square are congruent.  
C The area of all four right triangles sums to \( 2ab \).  
D The area of a triangle is \( \frac{1}{2}ac \).

33 What is the amplitude and period for the function \( y = 3.5 \sin \left( \frac{1}{3} \theta \right) \)?

A amplitude: 3.5  
B period: \( 3\pi \)  
C amplitude: 3.5  
D period: \( 6\pi \)  
E amplitude: 7.0  
F period: \( 3\pi \)  
G amplitude: 7.0  
H period: \( 6\pi \)
34 Which system of inequalities is represented by the graph below?

\[
\begin{align*}
\begin{cases}
 x - 2y &< 3 \\
 3x + y &\leq 4
\end{cases} & \quad \begin{cases}
 x - 2y &< 3 \\
 3x + y &\geq 4
\end{cases} \\
\begin{cases}
 x - 2y &> 3 \\
 3x + y &\geq 4
\end{cases} & \quad \begin{cases}
 x - 2y &> 3 \\
 3x + y &\leq 4
\end{cases}
\end{align*}
\]

35 Arrowhead Monument in Old Fort, North Carolina, is 30 feet tall. A laser measurement device placed on the ground forms the right triangle shown below.

**Approximately** how far is the laser measurement device from the monument?

\[
\begin{align*}
\text{A} & \quad 30.7 \text{ ft} \\
\text{B} & \quad 31.5 \text{ ft} \\
\text{C} & \quad 33.5 \text{ ft} \\
\text{D} & \quad 35.4 \text{ ft}
\end{align*}
\]

36 The number of North Carolina votes for the winning United States presidential candidate are shown in the table below.

<table>
<thead>
<tr>
<th>Years since 1980</th>
<th>Votes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>915,018</td>
</tr>
<tr>
<td>4</td>
<td>1,346,481</td>
</tr>
<tr>
<td>8</td>
<td>1,237,258</td>
</tr>
<tr>
<td>12</td>
<td>1,114,042</td>
</tr>
<tr>
<td>16</td>
<td>1,107,849</td>
</tr>
<tr>
<td>20</td>
<td>1,631,163</td>
</tr>
<tr>
<td>24</td>
<td>1,961,166</td>
</tr>
<tr>
<td>28</td>
<td>2,142,651</td>
</tr>
</tbody>
</table>

Using the least-squares regression line for the data, what will be the **approximate** number of North Carolina votes for the winning candidate in the 2012 presidential election?

\[
\begin{align*}
\text{A} & \quad 1,970,000 \text{ votes} \\
\text{B} & \quad 2,120,000 \text{ votes} \\
\text{C} & \quad 2,270,000 \text{ votes} \\
\text{D} & \quad 2,420,000 \text{ votes}
\end{align*}
\]

37 Which function is the inverse of \( f(x) = 7x - 10 \)?

\[
\begin{align*}
\text{A} & \quad f^{-1}(x) = \frac{1}{7}x + 10 \\
\text{B} & \quad f^{-1}(x) = \frac{1}{7}x + \frac{10}{7} \\
\text{C} & \quad f^{-1}(x) = -7x - 10 \\
\text{D} & \quad f^{-1}(x) = 7x + 10
\end{align*}
\]
38. Tracy and Orazio have two similar wooden blocks as shown below.

If the volume of Tracy’s block is 45 cubic centimeters, what is the volume of Orazio’s block?

- A) 90 cm³
- B) 180 cm³
- C) 270 cm³
- D) 360 cm³

39. The table below displays several points of a logarithmic function.

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>27</td>
<td>3</td>
</tr>
<tr>
<td>81</td>
<td>4</td>
</tr>
<tr>
<td>243</td>
<td>5</td>
</tr>
</tbody>
</table>

Which exponential function is an inverse of the function in the table?

- A) \( f^{-1}(x) = x^3 \)
- B) \( f^{-1}(x) = 3^x \)
- C) \( f^{-1}(x) = (x + 3)^3 \)
- D) \( f^{-1}(x) = 3^{x+1} \)

40. Which of the following expressions is equivalent to \( \frac{32^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}}{18^{\frac{1}{2}}} \)?

- A) \( \frac{4}{9} \)
- B) \( \frac{8\sqrt{2}}{3} \)
- C) \( \frac{4\sqrt{2}}{3} \)
- D) \( 4\sqrt{2} \)

41. Ray RZ bisects \( \angle QRS \). What statement can be inferred based on the given information?

- A) \( \angle QRZ \equiv \angle QRS \)
- B) \( m\angle QRS = 90° \)
- C) \( m\angle QRZ < m\angle QRS \)
- D) \( \angle QRZ \) and \( \angle ZRS \) are acute angles.

42. What is \( 81^{\frac{3}{4}} \) in simplest form?

- A) 27
- B) \( \sqrt[4]{27} \)
- C) 78
- D) 108
43 In Euclidean geometry, which diagram could be used to provide a counterexample to the conjecture below?

Through any three points, there exists exactly one plane.

A

B

C

D

45 Lola records survey responses in the pie chart shown below. The pie chart has a diameter of 40 inches, and it is divided into 3 sections.

What is the approximate length of the arc intercepted by the Pizza section of the chart?

A 29.7 in.  C 61.1 in.
B 34.9 in.  D 122.2 in.

46 North Carolina’s population has been increasing exponentially since 2000. The growth of the population \( P \) can be modeled according to the equation \( P = 8,046,500(1.0146)^t \), where \( t \) represents the number of years since 2000. Using the equation, predict how many years it will take for the population of North Carolina to reach 10,000,000.

A 2  C 15
B 7  D 86
The radius of the tire on Maggie’s bike is 12 inches. A point on the edge of the tire is at the bottom, touching the ground. After she rides a few feet, that same spot is $6\sqrt{2}$ inches above the ground. Which description best represents the rotation of the tire?

A 675° clockwise rotation  
B 690° clockwise rotation  
C 750° clockwise rotation  
D 780° clockwise rotation

Two functions, $g$ and $h$, are inverses of each other. If $x$ is a real number, for what values of $x$ is the equation $g(h(x)) = x$ true?

A all values of $x$  
B only some values of $x$  
C no values of $x$  
D impossible to determine

Which function has a graph with ends that both extend up?

A $f(x) = 3x^4$  
B $f(x) = 4x^3$  
C $f(x) = 4x^3$  
D $f(x) = 3x^4$

Maria records her January heating costs for ten years in the table below.

<table>
<thead>
<tr>
<th>Years Since 1998</th>
<th>Cost ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>163.00</td>
</tr>
<tr>
<td>1</td>
<td>166.60</td>
</tr>
<tr>
<td>2</td>
<td>172.20</td>
</tr>
<tr>
<td>3</td>
<td>177.10</td>
</tr>
<tr>
<td>4</td>
<td>179.88</td>
</tr>
<tr>
<td>5</td>
<td>183.96</td>
</tr>
<tr>
<td>6</td>
<td>188.90</td>
</tr>
<tr>
<td>7</td>
<td>195.30</td>
</tr>
<tr>
<td>8</td>
<td>201.60</td>
</tr>
<tr>
<td>9</td>
<td>207.34</td>
</tr>
<tr>
<td>10</td>
<td>215.30</td>
</tr>
</tbody>
</table>

She finds the least-squares regression line for the data in the table. For which of the following years is the absolute deviation from the least-squares regression line the greatest?

A 1998  
B 2003  
C 2004  
D 2008

A cylinder has a volume of 225 cubic inches. What is the volume of a sphere with the same height and diameter as the cylinder?

A 75 in$^3$  
B 112.5 in$^3$  
C 150 in$^3$  
D 337.5 in$^3$
52 Suppose WXYZ is graphed on the coordinate plane. Which argument could be used to prove WXYZ is a rectangle?

A Use the distance formula to verify both pairs of opposite sides are congruent.
B Use the slope formula to verify both pairs of opposite sides are parallel.
C Use the midpoint formula to verify the diagonals bisect each other.
D Use the distance formula to verify the diagonals are congruent.

53 Simplify the equation below. What restrictions must be placed on x?

\[ y = \frac{x^2 + 2x - 15}{x^2 - 25} \]

A \( y = \frac{x - 3}{x + 5} \);
   \( x \neq -5 \) and \( x \neq 5 \)
B \( y = \frac{x - 3}{x + 5} \);
   \( x \neq -5 \), \( x \neq 3 \), and \( x \neq 5 \)
C \( y = \frac{x - 3}{x - 5} \);
   \( x \neq -5 \) and \( x \neq 5 \)
D \( y = \frac{x - 3}{x - 5} \);
   \( x \neq -5 \), \( x \neq 3 \), and \( x \neq 5 \)

54 Given: \( \triangle LMN \sim \triangle HJK \)

What is the value of \( x \)?

A 10
B 12
C 14
D 16

55 Subtract \( \frac{y - 2}{6y} - \frac{4y - 3}{4y^2} \).

A \(-\frac{3y + 1}{24y^2}\)
B \(\frac{y^2 - 8y + 9}{6y^2}\)
C \(\frac{2y^2 - 16y - 9}{12y^2}\)
D \(\frac{2y^2 - 16y + 9}{12y^2}\)
56 In the triangle below, which equation could be used to find the value of \( x \)?

\[
\begin{align*}
\text{A} & \quad x = 18 \cos 65^\circ \\
\text{B} & \quad x = 18 \sin 65^\circ \\
\text{C} & \quad x = \frac{18}{\cos 65^\circ} \\
\text{D} & \quad x = \frac{18}{\sin 65^\circ}
\end{align*}
\]

57 A matrix is used to represent the distances between each of the following cities: Asheville, Raleigh, Charlotte, Greensboro, and Durham. Which statement best describes the matrix?

\[
\begin{align*}
\text{A} & \quad \text{The matrix has 5 rows and 5 columns, with 25 nonzero entries.} \\
\text{B} & \quad \text{The matrix has 5 rows and 5 columns, with 20 nonzero entries.} \\
\text{C} & \quad \text{The matrix has either 5 rows and 2 columns, or 2 rows and 5 columns, with 10 nonzero entries.} \\
\text{D} & \quad \text{The matrix has either 5 rows and 4 columns, or 4 rows and 5 columns, with 20 nonzero entries.}
\end{align*}
\]

58 In the figure shown below, what additional information is needed to prove the triangles are similar?

\[
\begin{align*}
\text{A} & \quad KL = 4 \text{ in.} \\
\text{B} & \quad KL = 25 \text{ in.} \\
\text{C} & \quad JK = 4 \text{ in.} \\
\text{D} & \quad JK = 7 \text{ in.}
\end{align*}
\]

59 What is the \( x \)-intercept of the graph of \( f(x) = \sqrt{x} - 1 \)?

\[
\begin{align*}
\text{A} & \quad (1, 0) \\
\text{B} & \quad (0, 0) \\
\text{C} & \quad (-1, 0) \\
\text{D} & \quad (0, -1)
\end{align*}
\]

60 If \( \log_{10} x = 5 \), what is the value of \( x \)?

\[
\begin{align*}
\text{A} & \quad x = \frac{1}{100,000} \\
\text{B} & \quad x = \frac{1}{50} \\
\text{C} & \quad x = 50 \\
\text{D} & \quad x = 100,000
\end{align*}
\]
61. In the figure below, the base areas and heights of the solids are equal. The volume of the pyramid is 562 cubic feet.

What is the volume of the prism?
A. 187 ft$^3$
B. 281 ft$^3$
C. 1124 ft$^3$
D. 1686 ft$^3$

62. The table below shows data for the heights of six students and the number of minutes they exercise each week.

<table>
<thead>
<tr>
<th>Height (inches)</th>
<th>Time (minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>55</td>
<td>130</td>
</tr>
<tr>
<td>61</td>
<td>20</td>
</tr>
<tr>
<td>70</td>
<td>160</td>
</tr>
<tr>
<td>63</td>
<td>240</td>
</tr>
<tr>
<td>73</td>
<td>100</td>
</tr>
<tr>
<td>68</td>
<td>90</td>
</tr>
</tbody>
</table>

Using the correlation coefficient $r$ which statement best describes the relationship between height and exercise time?
A. $r \approx 0.0$, there is no relationship
B. $r \approx 0.25$, there is a weak relationship
C. $r \approx 0.5$, there is a moderate relationship
D. $r \approx 0.75$, there is a strong relationship

63. The vertex-edge graph below shows the approximate number of miles between rest stops along several bike paths.

What is the distance of the shortest route between the East and West exits?
A. 41 miles
B. 42 miles
C. 43 miles
D. 45 miles

64. The function $f(x) = 2(x + 1)^2 - 5$ is transformed to $g(x) = -2(x + 1)^2 + 5$ in the coordinate plane. Which statement best describes the transformation?
A. The graph of $f(x) = 2(x + 1)^2 - 5$ is reflected across the x-axis.
B. The graph of $f(x) = 2(x + 1)^2 - 5$ is reflected across the x-axis and translated 10 units down.
C. The graph of $f(x) = 2(x + 1)^2 - 5$ is reflected across the x-axis and translated 10 units up.
D. The graph of $f(x) = 2(x + 1)^2 - 5$ is reflected across the y-axis and translated 10 units right.
65. The graph below shows the least-squares regression line for data points relating a person’s weight and the number of months the person has been on a restricted diet.

Using the mean absolute deviation from the least-squares regression line, which statement best describes the fit of the linear model for the data?

A. The mean absolute deviation is about 1 pound, so the linear model fits the data.
B. The mean absolute deviation is about 3.8 pounds, so the linear model fits the data.
C. The mean absolute deviation is about 14.9 pounds, so the linear model does not fit the data.
D. The mean absolute deviation is about 27 pounds, so the linear model does not fit the data.

67. In the figure below, H bisects $\overline{GY}$ and $\overline{JX}$.

Laura writes the argument below to prove $\angle JGH \cong \angle XYH$. What is the missing step in Laura’s argument?

i. $H$ bisects $\overline{GY}$ and $\overline{JX}$, so $\overline{GH} \cong \overline{HY}$ and $\overline{JH} \cong \overline{HX}$.

ii. $\angle GHJ$ is vertical to $\angle YHX$, so $\angle GHJ \cong \angle YHX$.

iii. ?

iv. Therefore, $\angle JGH \cong \angle XYH$.

A. $\overline{GJ} \parallel \overline{XY}$, so $\angle JGH \cong \angle XYH$ are alternate interior angles.

B. $\overline{GJ} \cong \overline{XY}$, $\overline{GH} \cong \overline{HY}$ and $\overline{JH} \cong \overline{HX}$, so $\triangle GHJ \cong \triangle YHX$ by SSS.

C. $\overline{GY} \perp \overline{JX}$, so $\triangle GHJ \cong \triangle YHX$ by LL.

D. $\angle GHJ \cong \angle YHX$, $\overline{GH} \cong \overline{HY}$, and $\overline{JH} \cong \overline{HX}$, so $\triangle GHJ \cong \triangle YHX$ by SAS.

66. What is the horizontal asymptote of the graph of $f(x) = -2x^3 + 4$?

A. $y = 4$
B. $y = 3$
C. $y = 0$
D. $y = -2$
68. If a point is randomly chosen inside the square, what is the probability that it is **outside** of the gray circle?

[Diagram: Square with a shaded circle]

A. 0.21  
B. 0.32  
C. 0.68  
D. 0.79

69. Triangle $EFG$ is transformed into triangle $E'F'G'$ as shown below.

[Diagram: Transformed triangles $EFG$ and $E'F'G'$]

The coordinate matrix for $EFG$ is multiplied by matrix $T$, resulting in the coordinate matrix for $E'F'G'$. What is matrix $T$?

A. $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  
B. $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  
C. $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  
D. $T = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

70. Joshua plans to invest $500 in a savings plan. Plan A guarantees $20 of interest per year. Plan B guarantees 4% interest compounded annually. Which functions represent the amount of money he would have on each plan after $x$ years?

A. Plan A: $f(x) = 500 + 20x$  
   Plan B: $f(x) = 500 + x^4$

B. Plan A: $f(x) = 500 + 20x$  
   Plan B: $f(x) = 500(1 + 0.04)^x$

C. Plan A: $f(x) = 500 + 20x^2$  
   Plan B: $f(x) = 500 + x^4$

D. Plan A: $f(x) = 500 + 20x^2$  
   Plan B: $f(x) = 500(1 + 0.04)^x$

71. Regina rolls a 6-sided die and randomly picks a token from a bag. There are 2 red tokens and 4 blue tokens in the bag. What is the probability of rolling an odd number and picking a red token?

A. $\frac{1}{2}$  
B. $\frac{1}{3}$  
C. $\frac{1}{4}$  
D. $\frac{1}{6}$
72 Although blood pressure devices no longer use mercury, blood pressure values are still reported in millimeters of mercury. For example, if blood pressure oscillates between 110 and 90 millimeters, the person’s blood pressure is 110 over 90. Rajiv’s blood pressure is modeled by the graph below, where \( m \) is millimeters and \( t \) is time in seconds.

What is Rajiv’s blood pressure?

A 100 over 40  
B 100 over 80  
C 120 over 40  
D 120 over 80

73 Which translation will transform the graph of \( y = x^2 \) into the graph of \( y = (x - 1)^2 + 2 \)?

A 2 units right  
B 2 units up and 1 unit right  
C 2 units up and 1 unit left  
D 1 unit up and 2 units right

74 Jack reads the following statement: “The student will earn his driver’s license if he passes the written test and the driving test.” Then, he creates the truth table shown below.

<table>
<thead>
<tr>
<th>Passes written test</th>
<th>Passes driving test</th>
<th>License?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Yes</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>No</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>No</td>
<td>No</td>
<td></td>
</tr>
</tbody>
</table>

How many times should Jack write “Yes” in the License column?

A 1  
B 2  
C 3  
D 4

75 The price of a first-class United States postage stamp is graphed on a coordinate plane. The x-axis represents the number of years since 1932, and the y-axis represents the price of the stamp. What type of function is most likely represented by the graph?

A an absolute value function  
B a quadratic function  
C an exponential function  
D a step function
76 Elaine has a scarf in the shape of the triangle shown below.

What is the value of $x$?

A $8$
B $8\sqrt{2}$
C $16$
D $16\sqrt{2}$

77 Given: Polygon $LMNPQ \sim VWXYZ$.

What is $m\angle W$?

A $20^\circ$
B $80^\circ$
C $100^\circ$
D $120^\circ$

78 Juniors and seniors at a high school will choose one of two restaurants for a school banquet. Matrix $C$ shows the costs of meals at each restaurant. Matrix $S$ shows the number of students that select each type of meal.

$$C = \begin{bmatrix} 8.50 & 9.25 & 7.70 \\ 11 & 12.50 & 10.25 \end{bmatrix}$$

$$S = \begin{bmatrix} 162 & 143 \\ 143 & 121 \\ 96 & 77 \end{bmatrix}$$

Matrix $P$ is the product of $C$ and $S$, as shown below.

$$P = C \times S = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Which element in matrix $P$ represents the total cost of meals for juniors at Restaurant 2?

A $P_{11}$
B $P_{12}$
C $P_{21}$
D $P_{22}$
The height of the tide in Cape Hatteras, North Carolina, is modeled in the graph below, where \( h \) represents height in feet and \( t \) represents time in hours.

Which statement best describes the height of the tide as modeled by the graph?

A. The tide peaks about every 3 hours and ranges from 0 to 3 feet.
B. The tide peaks about every 3 hours and ranges from 0.1 to 3.5 feet.
C. The tide peaks about every 12.5 hours and ranges from 0 to 3 feet.
D. The tide peaks about every 12.5 hours and ranges from 0.1 to 3.5 feet.

Barry drew the weighted graph below to represent a training program that involves 11 courses. The edges are labeled with the time, in hours, it will take to complete each course.

What is the critical path?

A. E, G, L, P
B. E, H, L, P
C. E, F, K, M, P
D. E, F, K, N, O, P